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DESIGN OF SAMPLED-DATA SYSTEMS BY EXTENSION OF CONVENTIONAL TECHNIQUES

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FOREWORD

A sampled-data control system is a linear system; its sampling device is the only element not found in conventional servomechanisms. The sampling device may be represented by an impulse modulator which can be described simply and conveniently in either the time domain or the frequency domain. By using the impulse modulator equivalent, one can analyze the whole sampled-data system either in the time domain or in the frequency domain by methods similar to those used on conventional systems. This analysis has already been described[1] in principle, and the formal techniques of analysis have been laid out. The great similarity between the analysis of sampled-data systems and that of conventional systems permits the translation of many techniques which have been used on conventional-system analysis into terms which apply to sampled-data systems. The function of this report is to describe the translated techniques in detail. The report is divided into three parts: (1) a description of signals and operations in sampled-data system components, (2) techniques for system studies using component characteristics, and (3) use of simplified versions of the analysis techniques described in design procedures.

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[1] Linvill, William K., Sampled-Data Control Systems Studied Through Comparison of Sampling with Amplitude Modulation, AIEE TRANSACTIONS, Volume 70, 1951.

ABSTRACT

The main advantage from realizing the similarity between continuous-data systems and sampled-data systems comes in design problems. System design is the art of compromising between what is desired in a certain system and what is physically realizable. Whereas a specific analysis problem has a unique solution, very often no specific design problem exists for a given situation and even if it did, there would exist many solutions rather than one. Accordingly, the technique of system design is usually a matter of cut-and-try wherein certain designs are visualized, analyzed, and modified. To be workable this cut-and-try process must have short steps based on simple approximate analysis procedures.

In this report analysis is discussed before design. First, the components in a sampled-data system are described both in the time and frequency domains. Second, techniques for flow graph (block diagram) manipulations are illustrated for reducing a complicated block diagram to a simple cascaded equivalent. Third, responses of a cascade of sampled-data system components are characterized by pole-zero locations and partial-fraction expansions in the s-plane, the ϵ^{-sT}-plane and the $\epsilon^{-sT/2}$-plane. The use of error coefficients and root-locus techniques are illustrated. Finally, design techniques are discussed after the analytical techniques are presented.

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1. DESCRIPTION OF SIGNALS AND OPERATIONS IN SAMPLED-DATA SYSTEM COMPONENTS

The key to the system of analysis used here is to compare sampling
with impulse modulation. Having made this comparison, one can represent
all elements in a sampled-data system by simple equivalent blocks and can
manipulate these blocks in the manner commonly used in the study of
continuous-data systems. The impulse modulator idea has been presented
in earlier papers,[1,2] but the function of this report is to show how it
is used in system design. The following paragraph presents a short summary
of the characteristics of all the fundamental types of linear components
which appear in sampled-data control systems. The remainder of Section 1.1
shows in a detailed fashion how the response charateristics of these components
are described in both the frequency domain and the time domain.

1.1 The Elementary Components in Sampled-Data Systems

The linear components in a sampled-data system are of four
kinds: (1) the impulse modulator, which has continuous input and sampled
output, (2) those conventional elements which have continuous inputs and
continuous outputs, (3) those components which have sampled inputs and
sampled outputs, and (4) those components which have sampled inputs and
continuous outputs.

1.11 The Impulse Modulator

In sampled-data systems there are two kinds of signals: sampled
and continuous. The process of sampling a continuous signal is the process
of selecting ordinates at regularly-spaced instants. The sampled signal
is in reality a set of selected ordinates, but it could be conceived as

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[2] Linvill, W.K. and Salzer, J.M., Analysis of Control Systems Involving
Digital Computers (To be published in I.R.E. Proceedings)

the limiting case of a continuous wave obtained by modulating a carrier of
sharp, tall pulses by the wave to be sampled. There is a one-to-one
correspondence between the successive samples of the signal and the areas
of the successive impulses in the modulated wave. Figure 1 shows the
characteristics of the impulse modulator. Treating the sampled wave as
the limiting case of a continuous wave allows one to think in terms of
conventional time functions and their Laplace transforms.
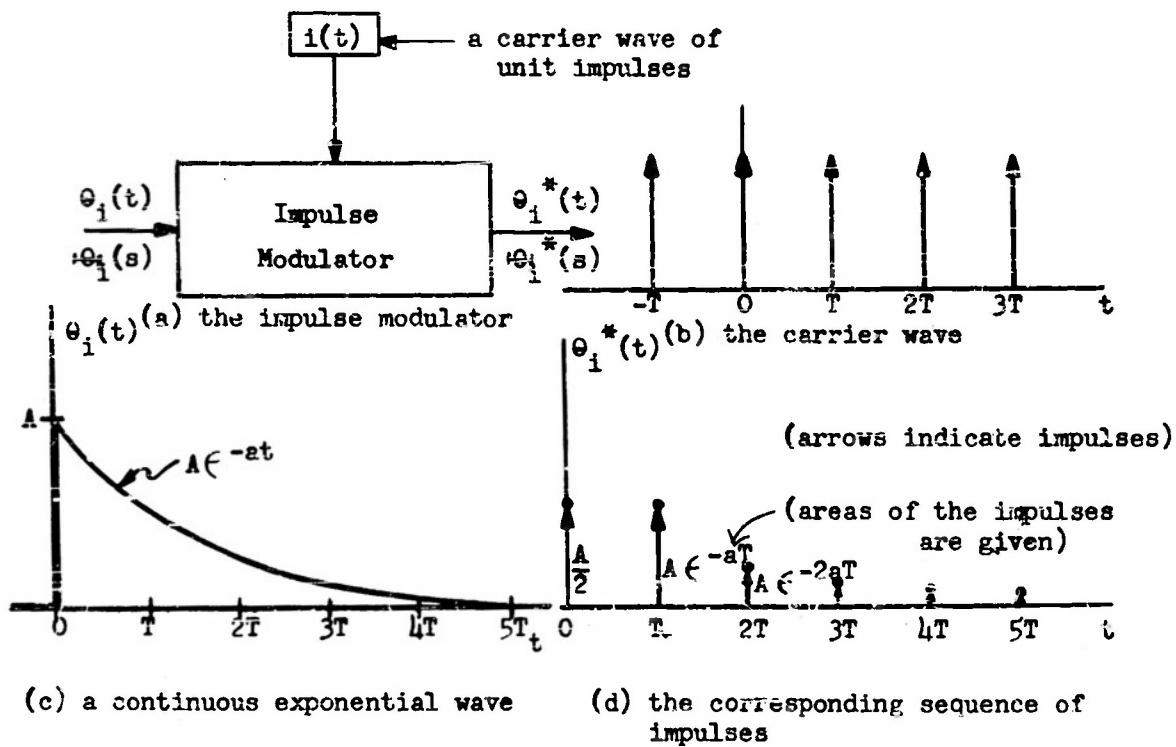
Fig. 1 Characteristics of the impulse modulator.

As Fig. 1 indicates, the carrier wave denoted by $i(t)$ is a string
of unit impulses separated by T seconds. The impulse modulator's output
is called $\theta_1^*(t)$ and is related to the input $\theta_i(t)$ by Eq. 1.**

$$\theta_i^*(t) = \theta_i(t) \times i(t) \tag{1}$$

The carrier wave of the impulse modulator can be resolved into
its various frequency components. When this is done it is represented
by an infinite series as in Eq. 2.

$$i(t) = \frac{1}{T}\sum_{k=-\infty}^{\infty} jk\Omega t, \text{ where} \tag{2}$$

$$\Omega = 2\pi/T.$$

Since the carrier of the impulse modulator has an infinite number of exponential
components instead of the two which correspond to a single sinusoidal carrier,
the impulse modulator output has many sidebands instead of only two as in the
case of a sinusoidal carrier. From this fact and the fact that superposition
of signals applies to the impulse modulator, the relation between the
impulse modulator input and its output in the frequency domain will be inferred.

If a single exponential component of frequency s and amplitude A
is applied to the input of the impulse modulator, then at the output there
appears a family of signals all of amplitude A/T and of frequencies s, $s+j\Omega$,
$s-j\Omega$, $s+j2\Omega$, $s-j2\Omega$, ... Thus, if $\theta_i(t) = A\epsilon^{-st}$, the output $\theta_i^*(t)$ is
defined by Eq. 3.

$$\theta_i^*(t) = \frac{A}{T}\sum_{n=-\infty}^{\infty} \epsilon^{(s+jn)t}. \tag{3}$$

** In this report impulse-modulated time functions will be distinguished
from continuous ones by the asterisk(*), thus $\theta_i^*(t)$ is the impulse
modulated $\theta_i(t)$.

If a more complicated signal is applied to the input of the impulse
modulator, the complex amplitudes of the various frequency components can
be represented by $\Theta_i(s)$. Each input component will result in an infinite
set of sidebands. From superposition Eq. 4 follows.

$$\Theta_i^*(s) = \frac{1}{T} \sum_{n=-\infty}^{\infty} \Theta_i(s+jn\Omega). \tag{4}$$

Equation 4 indicates that the transform of the impulse-modulated
output signal is periodic in the frequency domain, or that $\Theta_i^*(s) = \Theta_i^*(s+jm\Omega)$.
This fact is consistent with the fact that a periodic time function has
a sampled frequency function. Conversely, a sampled time function should
have a periodic frequency function, which it does.

Refer again to Fig. 1. The continuous time signal is an exponentially
decaying step which starts at t = 0. Because the carrier wave is a set of
narrow pulses which is even about the origin, the carrier pulse at t = 0
occurs at just the time when the discontinuity in $\Theta_i(t)$ occurs.[**] The
half of the carrier impulse occurring for t > 0 is multiplied by A.
The half of the t = 0 impulse occurring for t < 0 is multiplied by 0.
Hence the first output pulse area is $\frac{A}{2}$. Figure 1 shows that the impulse
modulator output is a collection of impulses each of which occurs at t=nT.
The transform of a unit impulse occurring at the origin is 1. The transform
of a unit impulse occurring at t=mT is ϵ^{-msT}. The transform of a whole
set of impulses is a sum of the individual transforms. Thus the transform
of the sequence of exponentially decaying impulses has the value given by
Eq. 5.

$$\left[\sum_{m=0}^{\infty} A\,\epsilon^{-amT}\epsilon^{-smT} - \frac{A}{2} \right] = A\left[\frac{1}{1-\epsilon^{-(s+a)T}} - \frac{1}{2} \right]. \tag{5}$$

[**]See Appendix A for a detailed discussion of this point.

For the same example the same result can be obtained from application of Eq. 4. For $\Theta_i(t)$ an exponentially decaying step $\Theta_i(s)$ is $\frac{A}{s+a}$. Whence

$$\Theta_i^*(s) = \frac{1}{T} \sum_{n=-\infty}^{\infty} \frac{A}{s+a+jn\Omega} = A\left[\frac{1}{1-\epsilon^{-(s+a)T}} - \frac{1}{2}\right]. \qquad (6)$$

All impulse modulator inputs having Laplace transforms which are rational functions in s can be resolved into partial fractions, and each term can be subjected to the summation of Eq. 6. Consequently, if $\Theta_i(s)$ is a rational function in s, $\Theta_i^*(s)$ is a rational function of ϵ^{-sT}. In summary, sampling is analogous to impulse modulation. The input wave of the impulse modulator can be related to the output in the time domain as in Eq. 1. The impulse modulator input and output can be related in the frequency domain as in Eq. 4. If $\Theta_i(s)$ is a rational function of s, $\Theta_i^*(s)$ is a rational function of ϵ^{-sT}.

1.12[**] Continuous Filters and Discrete Filters

Continuous filters are conventional filters which have continuous inputs and continuous outputs. They are familiar to engineers and are described in terms of their impulse response $h(t)$, or by their transfer function $K(s)$. Linear electric networks, linear amplifiers, and conventional linear servo components are all examples of continuous filters.

A discrete linear filter is a filter which accepts a sampled data input and transmits a sampled data output which is linearly related to the input. A discrete filter will accept input data only at the sampling instants and will transmit an output only at the sampling instants. Figure 2 shows

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[**] Precisely, these filters should be called continuous-signal filters and discrete-signal filters. The modifiers refer to the nature of the signal and not to the nature of the filter itself. Since all filters used will be of the lumped-parameter type and since frequent interchange between continuous and discrete signals will be made, all continuous-signal filters will be referred to as continuous filters and all discrete-signal filters will be referred to as discrete filters.

the parallelism between a continuous linear filter and a discrete linear filter. A digital computer with a linear program is equivalent to a linear discrete filter. Another example is a continuous filter with an impulse-modulated input and an impulse modulator at its output. This example is so prevalent that it is described in detail in Sec. 2.

$$\theta_o(t) = \int_0^\infty \theta_i(t-\tau)h(\tau)d\tau \qquad\qquad \theta_o^*(nT) = \sum_{k=0}^{\infty} \theta_i^* \big[(n-k)T\big] \; h(kT)$$

$$\theta_o(s) = \theta_i(s)K(s) \qquad\qquad\qquad \theta_o^*(s) = \theta_i^*(s)\,K^*(s)$$
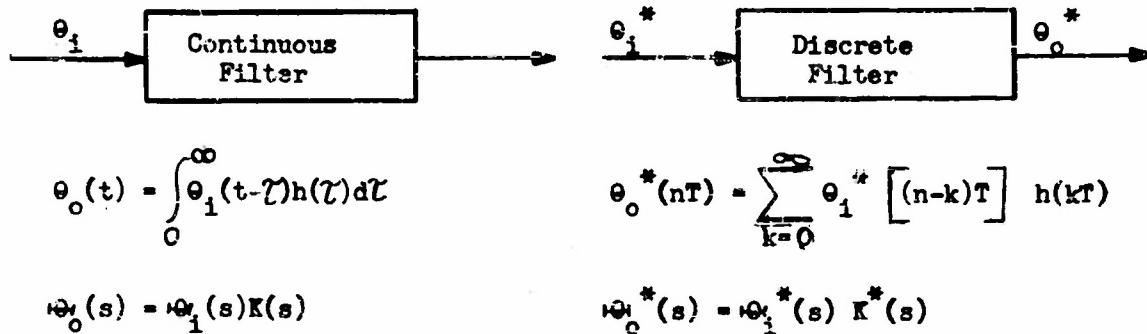
Fig. 2. Comparison between a continuous linear filter and a discrete linear filter.

The operation of a linear discrete filter can be described in the time domain by a difference equation such as Eq. 7.

$$\theta_o^*(nT) = h_o\theta_i^*(nT) + h_1\theta_i^*\big[(n-1)T\big] + h_2\theta_i^*\big[(n-2)T\big] + \text{----} \; h_k\theta_i^*\big[(n-k)T\big] + \text{---},$$

$$(7)$$

where:

$\theta_i^*(nT)$ = an impulse whose area is the output sample at $t = nT$.

$\theta_i^*(nT)$ = an impulse whose area is the input sample at $t = nT$.

h_k = the constants of the filter.

A discrete filter can be described equally well in the frequency domain by transfer functions which are rational functions of ϵ^{-sT}. The transfer function of the system is the ratio between the Laplace transform of the output and the Laplace transform of the input. Since if the Laplace transform of $\Theta_i^*(t)$ is $\Theta_i^*(s)$, then the Laplace transform of $\Theta_i^*(t-T)$ is $\Theta_i^*(s)\epsilon^{-sT}$, Eq. 7 can be expressed in the transform domain as in Eq. 8.

$$\Theta_o^*(s) = h_o\Theta_i^*(s) + h_1\epsilon^{-sT}\Theta_i^*(s) + h_2\epsilon^{-2sT}\Theta_i^*(s) + \ldots$$

$$h_k\epsilon^{-ksT}\Theta_i^*(s) + \ldots \tag{8}$$

The transfer function in this case becomes a power series in ϵ^{-sT}. For most practical cases the power series can be expressed in closed form and the transfer function may be expressed as a rational function of ϵ^{-sT} having only a limited number of poles in the ϵ^{-sT} plane. Techniques for expressing the transfer functions in various forms are described in Sections 1.211 and 1.212.

Note that both discrete and continuous filters can be described in either the time domain or the frequency domain. While a discrete filter can operate on only impulse-modulated inputs, a continuous filter can operate on both continuous inputs and on impulse-modulated inputs. In both cases the output is continuous. To find the Laplace transform of the output of the continuous filter when the input is impulse-modulated, merely multiply the transform of the input by the transfer function of the filter.

1.13 Summary of Characteristics of Elementary Components

All linear sampled-data systems may be represented by systems composed of a few elementary building blocks. The elements are the impulse modulator, continuous linear filters, and discrete filters. Both types

of filters are described by transfer functions in the frequency domain or
by unit signal responses in the time domain.   The impulse modulator is
linear, but is like an amplitude modulator rather than a filter, and
consequently cannot be described by a conventional transfer function.
It can be easily described in either the time domain or in the frequency
domain in a manner analogous to the way any amplitude modulator would be
described, as Eqs. 1 and 4 show..

1.2   Techniques for Handling Signals in Sampled-Data System Components

Thus far the types of components encountered in sampled-data
systems have been described.   The signals are either sampled or continuous.
This section is devoted to a review of how conventional techniques for
describing signals in continuous systems can be applied to the problem of
signals in sampled-data systems.

In sampled-data systems three classes of situations arise so far
as signal filters are concerned:

> (1) Continuous signals are applied to continuous filters,
>
> (2) Impulse-modulated signals are applied to discrete filters,
>
> (3) Impulse-modulated signals are applied to continuous filters.

The first case is the conventional problem. A continuous filter is described
precisely by a transfer function or by an impulse response. The filter's
precise response from any input may be obtained by multiplying the transfer
function by the input transform and then performing an inverse transformation.
Alternatively it can be obtained by convolving the impulse response with the
input time function.   Though these precise analysis techniques are always
applicable, a system designer is usually more interested in the <u>nature</u>
of the responses to each of several typical inputs than in the exact form
of any one response; consequently. he usually describes the response

characteristics in less precise but more practically meaningful form than the
exact transfer function provides. Two approximate ways are often used for
describing response characteristics of conventional continuous systems:

    (1) in terms of the amplitude-phase characteristics of the transfer function,

    (2) in terms of the nature of the impulse response.

       The form of the amplitude-phase curves of the transfer function
allows one to make certain generalizations about the nature of the response.
If the amplitude is approximately constant within the pass-band and if the
phase is linear, then the response will be satisfactory for all signals
whose frequency band-width is less than the cut-off frequency. The rise-time
of the response to a step is approximately equal to one-half the reciprocal
of the band-width in cycles per second, and the delay is proportional to the
phase slope. How much deviation from constant amplitude and linear phase
characteristics can be tolerated cannot be clearly stated in general, but
rule-of-thumb allows a system transfer function to have a resonant peak
in the response of about 30 present above the average value in the pass
band ($\dot{M}p = 1.3$)[3]. For more precise conditions the engineer usually makes
exact calculation of the transient responses.

       The superposition integral generalizes the system impulse response
to describe the system response to any input. Though the precise form of
the impulse response is not important, its nature is most important. Of
importance is the duration of the impulse response and the degree of
oscillation in it. The nature of the impulse response is easily correlated
with the poles and residues of the transfer function. Many practical systems
have only one pair of dominant poles, which usually are complex conjugates.
The amount and duration of oscillation in the impulse response is usually
determined by the ratio of the real part of the dominant poles to their

[3] Brown, O.S., and Campbell, D.P., Principles of Servomechanisms, John Wiley
and Sons, Inc., New York 1948.

magnitude. This ratio is commonly called the damping ratio, ξ .[4] A value

of ξ in the neighborhood of 0.7 is usually satisfactory. Since most transfer

functions of conventional systems are rational functions, they can be

described in terms of pole-zero constellations. For continuous systems,

simple graphical procedures exist to relate the transient response to the

pole-zero pattern.[5,6] These procedures are not described here, but their

analogues are described in connection with sampled-data transient response

calculations. The preceding few sentences have summarized the techniques

for describing signal operations in continuous systems. The analogues to

these techniques will now be applied in discrete-signal systems.

1.21   Operations on Impulse-Modulated Signals by Discrete Filters

As was stated earlier, a linear system having sampled inputs and

sampled outputs usually has a transfer function which is a rational function

of ϵ^{-sT}. From such a transfer function it is always possible to calculate ·

precisely the output corresponding to a given input. For the sampled-data

system this problem is actually simpler than for the continuous case, but

it is not a problem usually encountered by engineers, so it is discussed

in detail.

1.211   Precise Description of Signals in Discrete Filters

In the same sense that precise description of response of continuous

filters can be obtained from their transfer functions and precise description

of the input signal, the same sort of description can be made of responses

of discrete filters. If the transfer function $D(\epsilon^{-sT})$ is a rational function

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[4] Ibid., p. 49.

[5] Guillemin, E.A., Mathematics of Circuit Analysis, John Wiley and Sons, Inc.
New York.

[6] Truxal, J.G., Servomechanism Synthesis through Pole-Zero Configurations,
Sc.D. Thesis, M.I.T., 1950.

of ϵ^{-sT}, it may be written as a ratio of two polynomials as in Eq. 9.

$$D(\epsilon^{-sT}) = \frac{\theta_o^*(s)}{\theta_i^*(s)} = \frac{a_o + a_1\epsilon^{-sT} + a_2\epsilon^{-2sT} + \cdots + a_m\epsilon^{-msT}}{1 + b_1\epsilon^{-sT} + b_2\epsilon^{-2sT} + \cdots + b_k\epsilon^{-ksT}} \quad (9)$$

From this transfer function $\theta_o^*(s)$ may be expressed in terms of $\theta_i^*(s)$ as in Eq. 10.

$$\theta_o^*(s) + b_1\theta_o^*(s)\epsilon^{-sT} + b_2\theta_o^*(s)\epsilon^{-2sT} + \cdots + b_k\theta_o^*\epsilon^{-ksT} =$$

$$a_o\theta_i^*(s) + a_1\theta_i^*(s)\epsilon^{-sT} + a_2\theta_i^*(s)\epsilon^{-2sT} + \cdots + a_m\theta_i^*(s)\epsilon^{-msT}. \quad (10)$$

Because of the fact that each term in Eq. 10 can be identified immediately as the transform of a time function, $\theta_o^*(nT)$ may be expressed in terms of $\theta_i^*(nT)$.

$$\theta_o^*(nT) + b_1\theta_o^*\left[(n-1)T\right] + b_2\theta_o^*\left[(n-2)T\right] + \cdots + b_k\theta_o^*\left[(n-k)T\right] =$$

$$a_o\theta_i^*(nT) + a_1\theta_i^*\left[(n-1)T\right] + a_2\theta_i^*\left[(n-2)T\right] + \cdots + a_m\theta_i^*\left[(n-m)T\right] \quad (11)$$

The relation in Eq. 11 holds for all values of n. Given a sequence of input samples, $\theta_i(nT)$, and a set of initial conditions of $\theta_o^*(nT)$ one can use Eq. 11 to find $\theta_o(nT)$ for all n-values after the solution is started. For a specific example, suppose that $\theta_i^*(nT)$ is defined for all values from n = -m to n → ∞ and that $\theta_o^*(nT)$ is defined for all values from n = -k to n = -1. The value of $\theta^*(nT)$ may be evaluated for n = 0 by Eq. 11. Using the computed value, the value of $\theta_o^*(0)$ and Eq. 11 again, one can find $\theta_o^*(T)$. By repeating the process one can develop the whole sequence of output samples which result from the given input. The nature of this solution is identical with the nature of the solution of a differential equation by successive integrations to obtain an integral equation. The number of initial conditions necessary is exactly equal to the number of

transfer function poles in the e^{-sT}-plane. These initial conditions correspond
to the constants of integration in the integral-equation case. Because
finite time delays and not differentials are involved, the solution of Eq. 11
is much simpler than that of an integral equation. For times soon after
the start of the transient, Eq. 11 is very convenient to use; however, it
would be quite tedious to obtain the output for a large value of n from this
equation. To get a picture of the nature of the transient for a longer time,
expand the transfer function of Eq. 9 into a partial fraction expansion as
in Eq. 12.

$$D(\epsilon^{-sT}) = \frac{\alpha_1}{\epsilon^{-sT}-\beta_1} + \frac{\alpha_2}{\epsilon^{-sT}-\beta_2} + \cdots + \frac{\alpha_k}{\epsilon^{-sT}-\beta_2} . \qquad (12)$$

$\Theta_o^*(s)$ may be related to $\Theta_i^*(s)$ as in Eq. 9.

$$\Theta_o^*(s) = \frac{\alpha_1 \Theta_i^*(s)}{\epsilon^{-sT}-\beta_1} + \frac{\alpha_2 \Theta_i^*(s)}{\epsilon^{-sT}-\beta_2} + \cdots + \frac{\alpha_k \Theta_i^*(s)}{\epsilon^{-sT}-\beta_k} . \qquad (13)$$

Equation 13 leads one to break up the output, $\Theta_o^*(s)$, into components,
each of which is related to one natural mode (one pole) of the system.
The system of Eq. 14 results.

$$\Theta_o^{*(1)}(s) = \frac{\alpha_1 \Theta_i^*(s)}{\epsilon^{-sT} - \beta_1} = \frac{\frac{\alpha_1}{\beta_1} \Theta_i^*(s)}{1 - \frac{1}{\beta_1} \epsilon^{-sT}} ,$$

$$\Theta_o^{*(1)}(s) = \frac{\alpha_2 \Theta_i^*(s)}{\epsilon^{-sT} - \beta_2} = \frac{\frac{\alpha_2}{\beta_2} \Theta_i^*(s)}{1 - \frac{1}{\beta_2} \epsilon^{-sT}} , \quad --- \qquad (14)$$

$$\Theta_o^{*(k)}(s) = \frac{\alpha_k \Theta_i^*(s)}{\epsilon^{-sT} - \beta_k} = \frac{\frac{\alpha_k}{\beta_k} \Theta_i^*(s)}{1 - \frac{1}{\beta_k} \epsilon^{-sT}} .$$

The equivalent set of time functions is given in Eq. 15.

$$\Theta_o^{*(1)}(nT) = -\frac{1}{\beta_1} \Theta_i^{*}(nT) + \frac{1}{\beta_1} \Theta_o^{*(1)}(n-1)T \quad,$$

$$\Theta_o^{*(2)}(nT) = -\frac{2}{\beta_2} \Theta_i^{*}(nT) + \frac{1}{\beta_2} \Theta_o^{*(2)}(n-1)T \quad, \;---- \qquad (15)$$

$$\Theta_c^{*(k)}(nT) = -\frac{k}{\beta_k} \Theta_i^{*}(nT) - \frac{1}{\beta_k} \Theta_o^{*(k)}(n-1)T \quad.$$

To determine initial conditions of all components of $\Theta_o^{*}(nT)$, the same
number of initial values of Θ_o^{*} are needed as when Eq. 11 is used.
The procedure for finding the initial conditions closely parallels the
procedure for getting initial conditions in the solution of conventional
differential equations. If all values of β have magnitude greater than
unity, all components of the output will ultimately die out with no
input. If any value of β has magnitude equal to 1, its component will
never die out regardless of what the input does. If any β exists having
a magnitude less than 1, the response will grow without limit. Other
maniuplations of the transfer function could be made in order to emphasize
specific aspects of the response, but these manipulations will not be
described in detail at this point.

This procedure for analyzing a system with sampled input and
sampled output is analogous to the operation in a continuous system of
precisely relating an output time function to an input time function.[*]

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[*] The general problem of relating transforms of sampled functions to
the sampled functions will be treated in Appendix B.

1.212 Approximate Characterization of Operation on Signals
in Discrete Filters

The nature of response characteristics of a discrete filter
may often be obtained from a study of the amplitude and phase of the
transfer function as functions of the complex frequency s. To see
physically the connection between the amplitude and phase description
of this system and that of a continuous signal system, refer to Fig. 3.
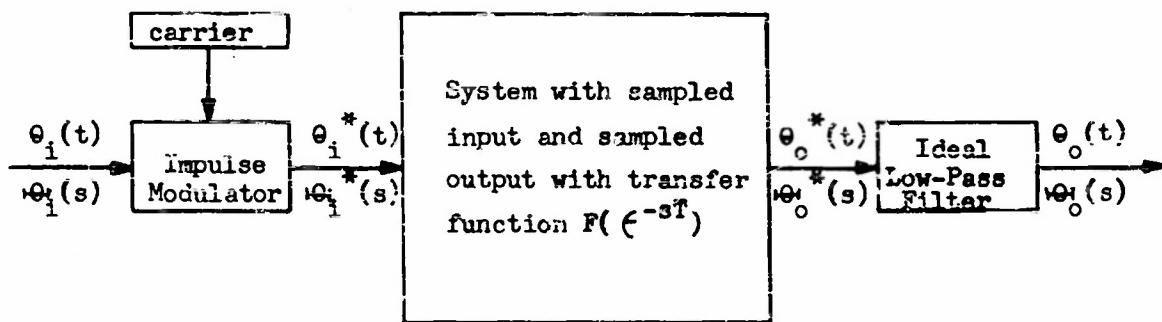
Fig. 3. A system with sampled input and sampled output fed by
input sampler and with an ideal low-pass filter at the
output to act as a desampler.

The signal $\theta_i(s)$ is a conventional continuous signal which may be described
in the frequency domain or in the time domain. The impulse modulator
samples the continuous signals and thereby adds the complementary signals
as indicated by Eq. 4 and illustrated by Fig. 4. As the signal $\theta_i^*(s)$ is
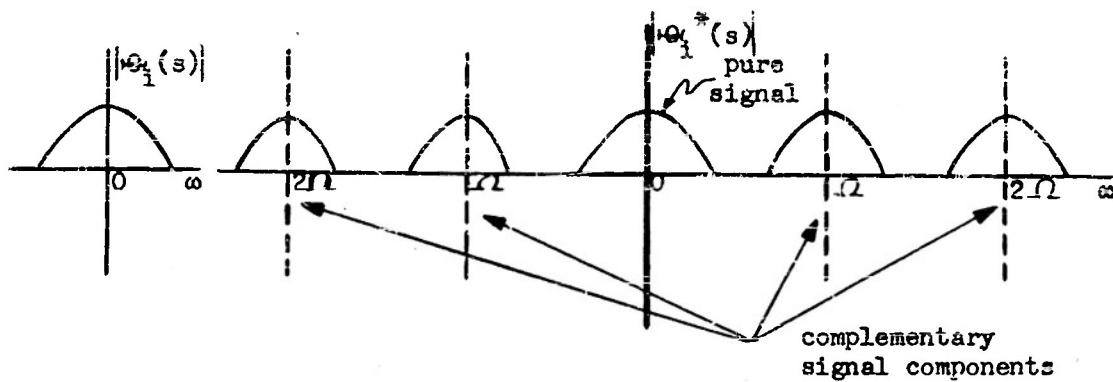
Fig. 4. Amplitude spectra of a continuous signal and of the
corresponding impulse-modulated signal.

fed into the system having a transfer function $D(\epsilon^{-sT})$, each component
of the signal is handled in precisely the same manner as each other component
because $D(\epsilon^{-sT})$ is periodic in frequency of period Ω. One can either
interpret the result in terms of the variable ϵ^{-sT}, in which case the
samples stand out, or in terms of the variable s, in which case the
frequency components of the corresponding unsampled signal stand out. If
the amplitude and phase of $D(\epsilon^{-sT})$ are plotted as a function of s for
$s = j\omega$, then one can visualize how each frequency component of each complementary
component of $\Theta_i^*(s)$ is operated upon. Since $D(\epsilon^{-sT})$ is a rational
function of ϵ^{-sT}, it is very easy to visualize its value as a quotient
of products of vectors in the ϵ^{-sT}-plane. Suppose $D(\epsilon^{-sT})$ has a
constellation of poles and zeros as shown in Fig. 5. The value of the
amplitude and phase functions is shown in Fig. 5b. The "pass band"
of the system is in the range from about $-j\Omega/4$ to $+j\Omega/4$ radians per second.

ϵ^{-sT}-plane

$s = j\frac{3\Omega}{4}$

locus of $s = j\omega$

$s = j\frac{\Omega}{2}$

$s = 0$

$\left[\epsilon^{-sT} - (1.1 + j0.3)\right]$

(ϵ^{-sT})

$s = j\frac{\Omega}{4}$

$\left[\epsilon^{-sT} - (1.1 - j0.3)\right]$

(a)

$|D|$

$-\Omega$ $-\frac{3\Omega}{4}$ $-\frac{\Omega}{2}$ $-\frac{\Omega}{T}$ 0 $\frac{\Omega}{T}$ $\frac{\Omega}{2}$ $\frac{3\Omega}{4}$ Ω ω

$\underline{/D}$

(b)
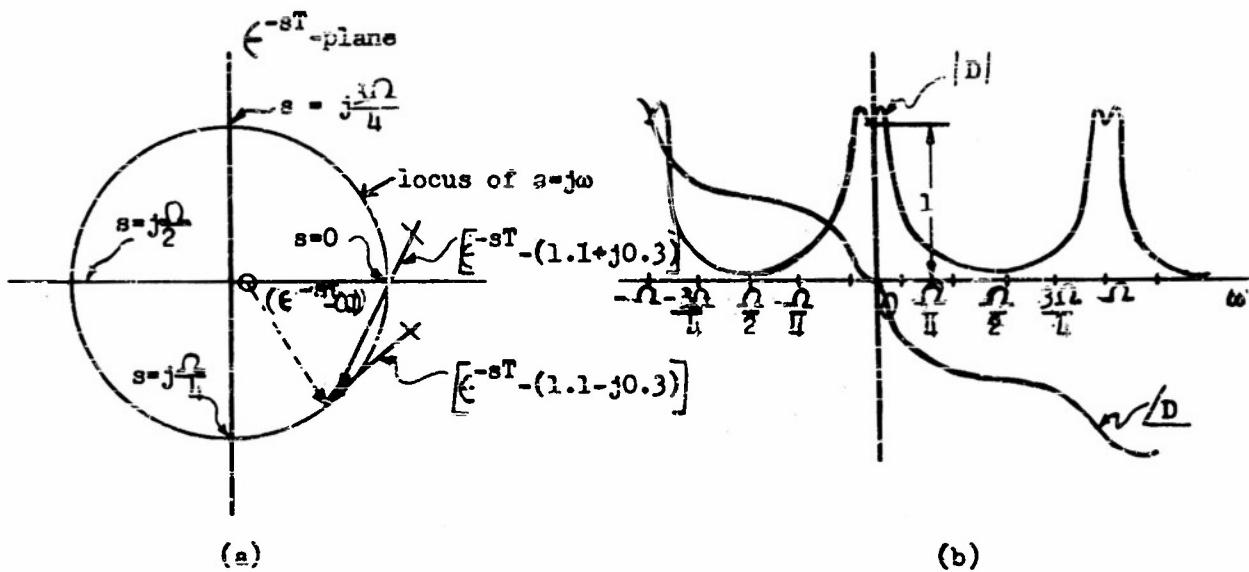
Fig. 5. The plot of the amplitude and phase of $D(\epsilon^{-sT})$ correlated with its constellation of poles and zeros for

$$D = \frac{\frac{1}{9}(\epsilon^{-sT} - 0.1)}{\left[\epsilon^{-sT} - (1.1 - j0.3)\right]\left[\epsilon^{-sT} - (1.1 + j0.3)\right]} .$$

The resonant peak occurs at the frequency for which the point on the unit circle lies nearest the pole of D, or at about $j\Omega/20$. At these points, also, the phase has the steepest slope.

To relate $\Theta_0^*(s)$ to $\Theta_1^*(s)$, merely multiply the components of $\Theta_1^*(s)$ whose magnitudes are indicated in Fig. 4 by the complex function D plotted in Fig. 5b. If the signal Θ_0^* is fed into an ideal low-pass filter which rejects all the complementary signals but passes the pure signal without delay or distortion, the overall result is the same as if

the unsampled input signal $\Theta_0(s)$ were fed into a continuous filter having a transfer function equal to D in the range from $-j\Omega/2$ to $+j\Omega/2$ and equal to zero outside this range.

The equivalent to the impulse response of a continuous system is the unit-sample response of a sampled-data system. The unit-sample response can be obtained by the technique of either Eq. 11 or Eq. 15. From a study of the poles and residues of $D(\epsilon^{-sT})$, one can deduce the form of the unit-sample response. Just as in the continuous-signal case, if the sampled-data system has two dominant poles (two which are much closer to the unit circle in the ϵ^{-sT}-plane than all other poles), one can characterize the nature of the response in terms of the ζ of these two poles. Figure 6

(a)   (b)
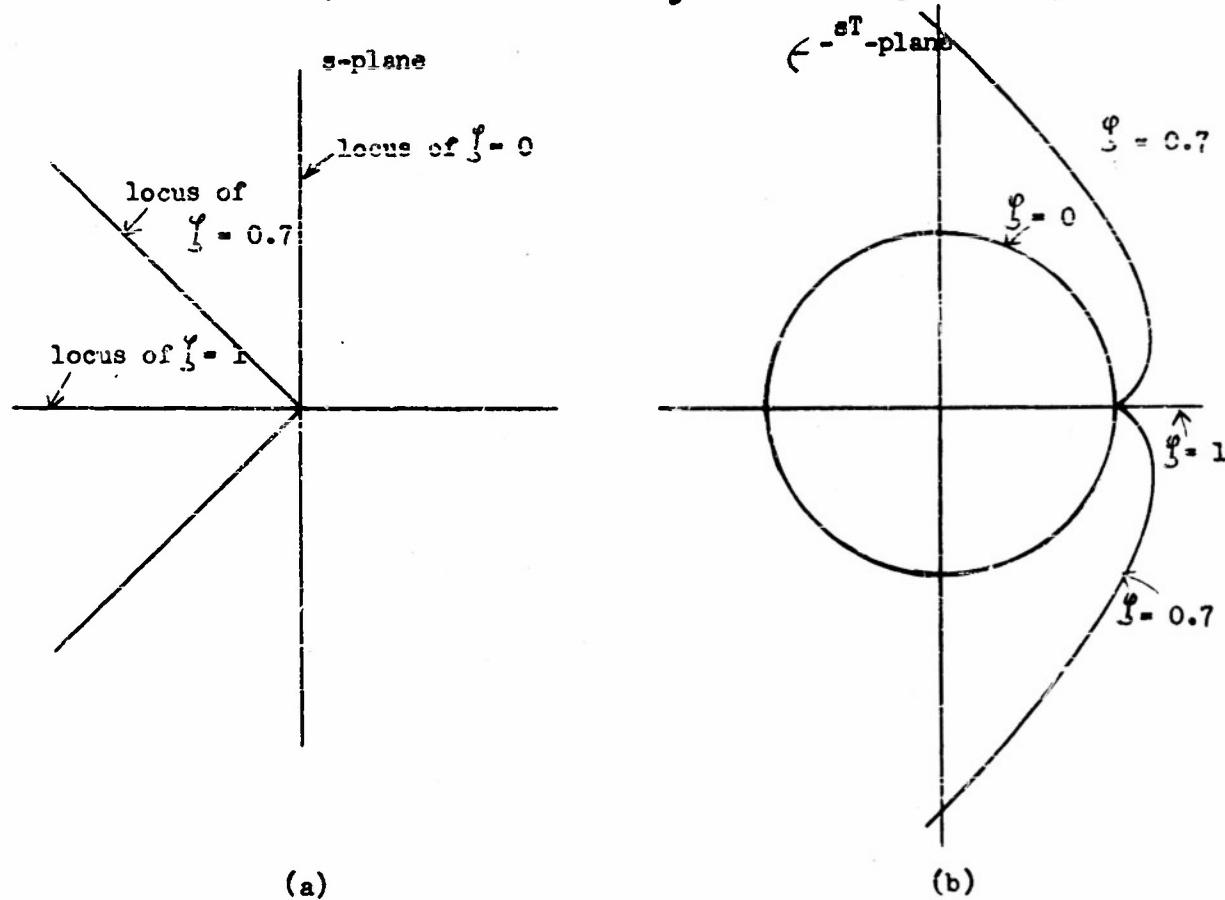
Fig. 6. Loci of constant ζ in the s-plane and their corresponding locations in the ϵ^{-sT}-plane.

shows loci of constant $\varphi \zeta$ in the ϵ^{-sT}-plane. The value of ζ has the same meaning for sampled systems as it does for continuous systems. It gives a measure of how fast an oscillatory transient dies out relative to its period of oscillation. For values of ζ much smaller than 0.7 the transients are highly oscillatory; for values of ζ much larger than 0.7 the transients are highly damped. If there are more than one pair of dominant poles, the impulse response can be calculated but cannot be predicted readily from pole positions because there are too many components.

2. RESPONSE OF CASCADED SYSTEMS

In the preceding section methods were given for finding the response of the three basic types of sampled-data system components - the modulator, continuous-signal filter and discrete-signal filter. In this section we treat the problem of finding the response of any cascaded sequence of such components.

Evidently the calculation of overall response of such a system could be done stepwise by methods already given, proceeding from component to component. However, such a procedure is more tedious than necessary since any combination of cascade components can be reduced to an equivalent simple form with the same response. The fundamental equivalence which permits this reduction is shown in Fig. 7. The reduction consists of replacing

(a)

$\theta_i^*(t)$ Continuous Filter $K(s)$ $\theta_A(t)$ Impulse Modulator $\theta_o^*(t)$

carrier

Sampled inputs only   A

Original System

(b)

$\theta_i^*(t)$ $K^*(s)$ $\theta_o^*(t)$
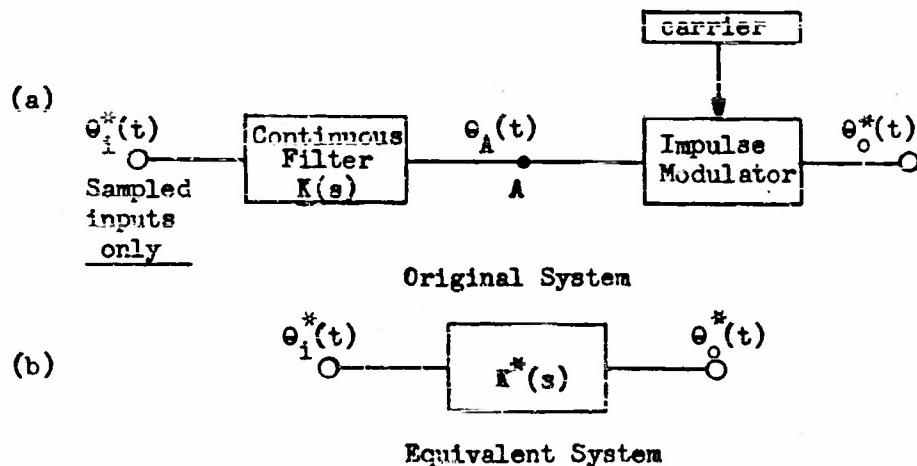
**Equivalent System**

Fig. 7. Reduction of a cascade system to a single element. the combination of a continuous filter followed by an impulse modulator by an equivalent discrete filter when the input signals are sampled.

The equivalence is derived by means of the following procedure. Apply a sampled input $\theta_i^*(t)$. The transform of the signal applied to the impulse modulator is then:

$$\Theta_A(s) = \Theta_i^*(s) \cdot K(s). \tag{16}$$

The output transform is then calculated from this result by the summation procedure of Eq. 4:

$$\Theta_o^*(s) = \frac{1}{T} \sum_{n=-\infty}^{+\infty} \Theta_A(s + jn\Omega) = \frac{1}{T} \sum_{n=-\infty}^{+\infty} K(s + jn\Omega)\Theta_i^*(s + jn\Omega). \tag{17}$$

But since $\Theta_i^*(t)$ is a purely sampled signal, $\Theta_i^*(s)$ is periodic, i.e.,

$\Theta_i^*(s) = \Theta_i^*(s + jn\Omega)$ for any integer n. Therefore

$$\Theta_o^*(s) = \Theta_i^*(s) \cdot \frac{1}{T} \sum_{n=-\infty}^{+\infty} K(s + jn\Omega) \tag{18}$$

so that the transfer function of an equivalent system is

$$K^*(s) = \frac{1}{T} \sum_{n=-\infty}^{+\infty} K(s + jn\Omega). \tag{19}$$

Since $\frac{1}{T} \sum_{n=-\infty}^{\infty} K(s + jn\Omega)$ is a function of s which is multiplied by

$\Theta_i^*(s)$ to give $\Theta_o^*(s)$, it is the transfer function of the discrete filter

made up of a continuous filter and an impulse modulator. Recall that the

components which accept sampled signals and deliver sampled signals have

transfer functions which are rational functions of ϵ^{-sT}. It is interesting

to note that if K(s) is a rational function of s, $\frac{1}{T} \sum_{n=-\infty}^{\infty} K(s + jn\Omega)$ is

a rational of ϵ^{-sT} obtained by resolving K into simple terms by a

partial fraction expansion and then applying the summation used in Eq. 6

to each term. Since one goes through the same process to relate $\Theta_i(s)$

to $\Theta_i^*(s)$ as to relate K(s) to $\frac{1}{T} \sum_{n=-\infty}^{\infty} K(s + jn\Omega)$, we shall refer to

$\frac{1}{T} \sum_{n=-\infty}^{\infty} K(s + jn\Omega)$ as $K^*(s)$.

2.1 Response of the $K_a MDK_b$ System

The above equivalence allows the immediate reduction of any cascaded combination of components to the standard form of a continuous filter (K_a) followed in succession by a modulator (M), a discrete filter (D), and a second continuous filter (K_b). (In degenerate forms some of these components may be missing.) The reduction proceeds as follows: If the cascade contains only one modulator, it is already in standard form. If it contains two modulators, between them there is a transition from sampled signals to continuous signals in preparation for a new sampling. Sampled signals are delivered to a continuous filter to be smoothed and are then remodulated. This smoothing-remodulating operation then can be combined in one equivalent discrete filter operation. Cascade arrangements with any number of modulators are treated by applying this reduction to one filter-modulator set at a time.

It is true that any sampled-data system with or without feedback can be reduced to a standard form containing nothing but parallel branches of the above basic $K_a MDK_b$ type. At this point we do not wish to go into any details of these reductions to equivalent forms. (See Section 2.2). Here it is sufficient to note that solution of the $K_a MDK_b$ system response problem in effect solves the general analysis problem, once suitable reduced equivalent systems are obtained.

2.11 An Exact Calculation of Response

Suppose an input signal $\theta_i(t)$ is applied to the $K_a MDK_b$ system as shown in Fig. 8a. We denote the transform of the input signal by $\theta_i(s)$ and apply similar notation for the output signal. But the same signal appears at point A if we think of $\theta_i(t)$ as being generated by a single unit impulse applied at time $t = 0$ to a filter whose transfer function is $\theta_i(s)$. Thus Fig. 8b shows a system whose unit impulse

a)

$\theta_i(t)$ Input $\boxed{K_a(s)}$ A $\boxed{M}$ $\boxed{D(e^{-sT})}$ $\boxed{K_b(s)}$ $\theta_o(t)$ Output

$\theta_i(s)$ $\theta_o(s)$

b) Single Impulse Input $\boxed{\theta_i(s)K_a(s)}$ A $\boxed{M}$ B $\boxed{D(e^{-sT})}$ $\boxed{K_b(s)}$ $\theta_o(t)$

$\theta_o(s)$

c) 1 $\boxed{(\theta_i K_a)^*}$ B $\boxed{D}$ $\boxed{K_b}$ $\theta_o(s)$
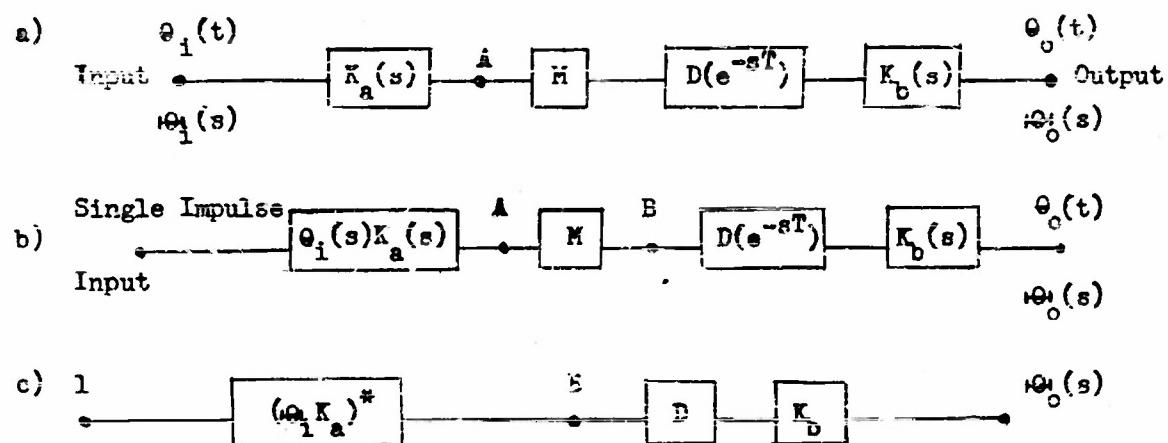
Fig. 8   Reduction of a cascade system.

response is the required $\theta_o(t)$. We now recognize the combination of a continuous signal filter followed by an impulse modulator to which a sampled signal is applied -- namely, the unit impulse. This combination is then reduced as shown in Fig. 8c to the equivalent discrete signal filter, $\frac{1}{T}\sum(\theta_i K_a)$, denoted conveniently by the starred notation $(\theta_i K_a)^*$. Finally, we determine the output transform by multiplying the input transform, 1, by the overall system transfer function. The result is $\theta_o = (\theta_i K_a)^* D K_b$, from which the output time function can be determined.

The difference between this procedure and the transform analysis of conventional systems lies in the fact that here the presence of the modulator and the multiple frequency translations which it introduces require the summation process or $\theta_i K_a$ in addition to a multiplication of component transfer functions. Another way to look at this situation is to view the modulator as a barrier. The barrier can be jumped by changing to a time-domain description, sampling the input to the modulator, deducing the transform of the samples, and finishing in the frequency domain. This time-domain viewpoint often affords an easy way to find $(\theta_i K_a)^*$ without actually summing.

The features of the basic $K_a MDK_b$ response calculations are illustrated in the example of Fig. 9. The $K_a MDK_b$ system of Fig. 5 may have been derived by reducing a more general system to the standard form. Its response to a step input applied at a sampling time will now be computed.

$$\theta_i(s) = \frac{1}{s} \quad \boxed{\begin{array}{c} K_a(s) \\ \frac{1}{s+a} \end{array}} \quad \boxed{M} \quad \boxed{\frac{D(e^{-sT})}{1-2e^{-sT}}} \quad \boxed{\frac{1}{s+b}} \quad \theta_o(s)$$
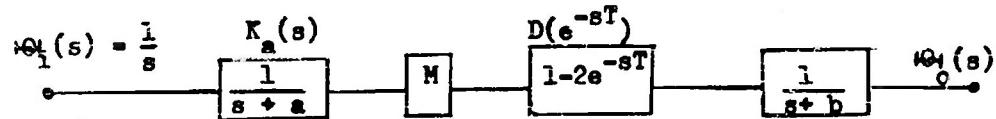
Fig. 9 A basic $K_a MDK_b$ system responding to a step

The signal applied to the modulator has the transform $\theta_i K_a = \frac{1}{s(s+a)}$. After modulation the signal has the transform $(\theta_i K_a)^*$ given by,

$$(\theta_i K_a)^* = \frac{1}{T} \sum_{n=-\infty}^{+\infty} \theta_i(s+jn\Omega) K_a(s+jn\Omega)$$

$$= \frac{1}{T} \sum_{n=-\infty}^{+\infty} \left[\frac{1}{(s+jn\Omega)(s+a+jn\Omega)} \right]$$

$$= \frac{1}{T} \sum_{n=-}^{} \left[\frac{\frac{1}{a}}{s+jn\Omega} - \frac{\frac{1}{a}}{s+a+jn\Omega} \right] \qquad (20)$$

$$= \frac{1}{a} \left[\frac{1}{1-e^{-sT}} - \frac{1}{1-e^{-(s+a)T}} \right]$$

$$= \frac{1-e^{-aT}}{a} \cdot \frac{e^{-sT}}{(1-e^{-sT})(1-e^{-aT}e^{-sT})}$$

Instead of summing directly we may change momentarily to the time domain to jump the modulator barrier. The applied signal has the transform

$\theta_i K_a = \dfrac{\frac{1}{a}}{s} - \dfrac{\frac{1}{a}}{s+a}$, which in partial fraction form indicates a step

and an exponential component of amplitudes $\frac{1}{a}$ and $-\frac{1}{a}$. The sequence of
samples from the step (assuming that the sampling occurs just after a
discontinuity) is just a sequence of samples all of amplitude $\frac{1}{a}$ with the
transform

$$(\theta_1 K_a)_1^* = \frac{1}{a}\left[1 + e^{-sT} + e^{-2sT} + \ldots\right] \tag{21}$$

The sequence of samples from the exponential similarly has the transform

$$(\theta_1 K_a)_2^* = -\frac{1}{a}\left[1 + e^{-aT}e^{-sT} + e^{-2aT}e^{-2sT} + \ldots\right] \tag{22}$$

Rolling up these expressions into closed form by inspection and adding, since
the modulation process is linear, we find the total modulator output

$$(\theta_1 K_a)^* = (\theta_1 K_a)_1^* + (\theta_1 K_a)_2^*$$

$$= \frac{1}{a}\cdot\frac{1}{1-e^{-sT}} - \frac{1}{a}\cdot\frac{1}{1-e^{-aT}e^{-sT}} \tag{23}$$

$$= \frac{1-e^{-aT}}{a}\cdot\frac{e^{-sT}}{(1-e^{-sT})(1-e^{-aT}e^{-sT})}$$

This answer is the same as that found previously by direct summation.

        To find the output signal transform, we now treat the remainder
of the system in conventional manner and multiply the remaining transfer
functions.

$$\theta_0(s) = (\theta_1 K_a)^*\, D \cdot K_b$$

$$= \frac{1-e^{-aT}}{a}\cdot\frac{e^{-sT}}{(1-e^{-sT})(1-e^{-aT}e^{-sT})}\cdot(1-2e^{-sT})\cdot\frac{1}{(s+b)}$$

$$= \frac{1-e^{-aT}}{a}\cdot\frac{e^{-sT}(1-2e^{-sT})}{(1-e^{-sT})(1-e^{-aT}e^{-sT})\,(s+b)} \tag{23a}$$

To get an overall picture of this response we plot the poles and zeros of $\Theta_o(s)$ in the s-plane, as in Fig. 10.
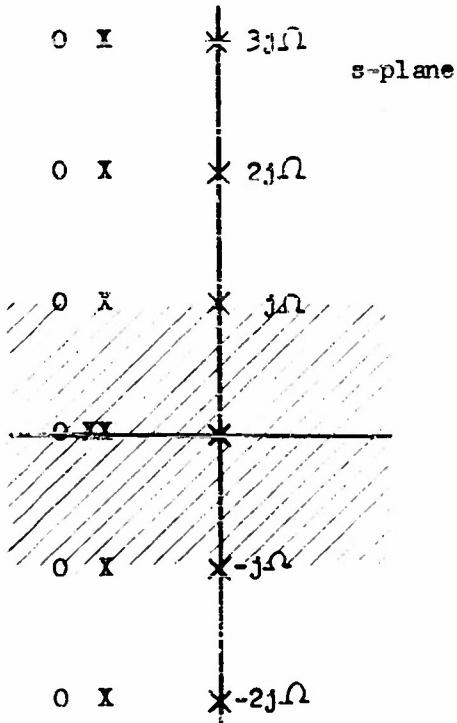
Fig. 10 Poles and zeros of $\Theta_o(s)$ in the s-plane.

In general, several infinite sequences of poles and zeros are obtained because of the factors of $\Theta_o(s)$ which are rational in e^{-sT}. In addition, there are factors rational in s which lead to a finite number of poles and zeros. The e^{-sT} factors yield a pattern of period $j\Omega$.

In this diagram various components of the output may be associated with the poles. The pole at the origin signifies that part of the output is a step function. The two other poles on the real axis represent two exponential transient components with different time-constants. These three poles together form a central group of relatively low-frequency response. It is for this part of the response that a sampled-data system is designed. The remaining poles represent that part of the response called the ripple.

The sequence of poles at $\pm j\Omega$, $\pm 2j\Omega$, ... represent a ripple component persisting in time, the steady-state ripple. The poles at $\pm j\Omega -a$, $\pm 2j\Omega -a$, ... represent a damped ripple component.

Suppose that we let $T \longrightarrow 0$ - i.e., $\Omega \longrightarrow \infty$. Then the sampled-data system approaches a continuous system. In Fig. 10 the $s = \pm j\Omega$ lines recede, leaving the central group of response poles in complete command of the s-plane. We look for our basic response poles inside the strip shaded in Fig. 10 between the lines $s = +j\Omega$ and $s = -j\Omega$.

A sketch of the pole-zero diagram of the output transform, then, immediately shows us the qualitative nature of the response. In addition it suggests the application of the procedure of numerical evaluation of the pole residues to build up the output response from its various natural modes. However, an annoying feature of such quantitative work is the occurrence of mixed expressions containing both s and e^{-sT}. In the Sec. 2.12 we show how one can arrive at an approximate determination of response which is simpler than the direct method above. The s-plane diagram has its major value in exhibiting an overall qualitative picture of the response. A sketch of the s-plane diagram would be a preliminary step in quantitative analysis.

2.12 An Approximate Calculation of Response

The approximate method for computing the response of the $K_a MDK_b$ system rests on the observation of the previous section that the response consists essentially of two parts, a "first order response" with components of low natural frequency and a ripple response of high-frequency components. Our approximate method will determine each of these effects separately in the following manner.

The first order response may be found by computing the system output only at the instants at which the impulse modulator samples its input. In other words, we inquire what the output is at $t = 0$, T, $2T$, $3T$... to a transient input applied at $t = 0$. This procedure is approximate only in the sense that we ignore what the output does between the sampling instants; the values of the output at these instants are to be determined accurately.

In order to determine the output of the $K_a MDK_b$ system, we connect to the output of the system a second impulse modulator which runs in phase with the first (see Fig. 11).

a)   original system

b)   modified system

c)   equivalent of modified system

$$\Theta_o^* = (\Theta_i K_a)^* DK_b^*$$
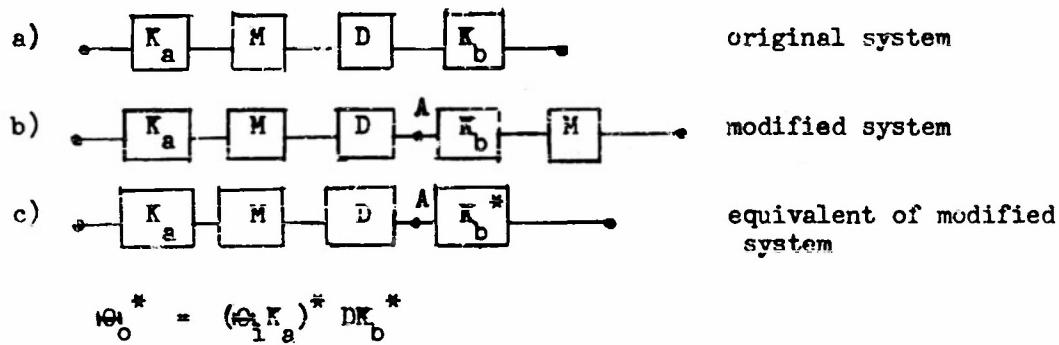
Fig. 11 Reduction of second modulator from a cascade.

Then the new output is a sequence of impulses whose areas are identified with the value of the original output at the sampling times. Since at point A and the new output terminal the signals are of sampled nature, an equivalent discrete signal filter may be found. Its transfer function is

$$K_b^* = \frac{1}{T} \sum K_b.$$

Now if we apply the results of the previous section in computing the transform of the sampled output when the input transform is given, we obtain immediately $\Theta_o^* = (\Theta_i K_a)^* DK_b^*$, where the * sign in the symbol Θ_o^* indicates that we are dealing with the sampled output. This result differs

from the previous one in that now θ_o^* is a rational function of e^{-sT} with no factors rational in s like those which occurred in the expression for θ_o. In the previous discussion the s-plane was used to sketch the poles and zeros of the response. Here, however, it is more useful to use the e^{-sT} plane. Furthermore, since θ_o^* is rational in e^{-sT}, only a finite number of poles and zeros appear in the e^{-sT} plane. As far as the sampled output is concerned, then, only a finite number of modes of vibration arise. Their magnitude may be found by either simple graphical measurements on the e^{-sT} plane pole-zero pattern or by breaking up θ_o^* analytically into partial fractions, each term of which represents a single mode of response.

The analytical procedure is illustrated in the following example. The diagram of the system to be investigated appears in Fig. 12. A unit step will be applied to the input. We require the output at the sampling instants.

$$\theta_i = \frac{1}{s}$$
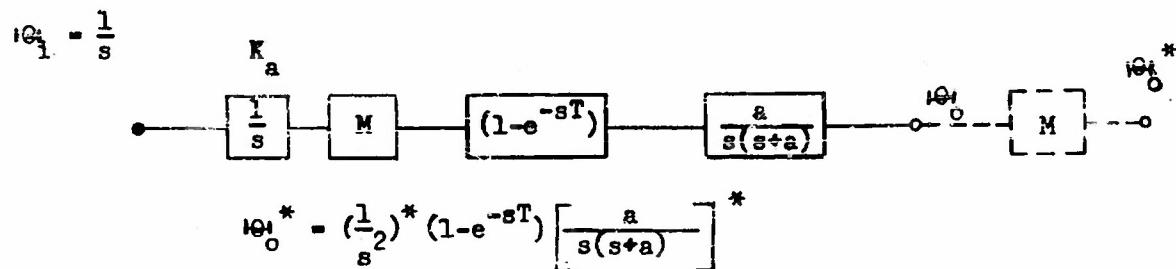
$$\theta_o^* = \left(\frac{1}{s^2}\right)^* (1-e^{-sT}) \left[\frac{a}{s(s+a)}\right]^*$$

Fig. 12 Process for finding sampled response

The complete output has the transform

$$\theta_o = \left(\frac{1}{s^2}\right)^* (1-e^{-sT}) \cdot \frac{a}{s(s+a)} \tag{24}$$

$$= \frac{Te^{-sT}}{(1-e^{-sT})^2} \cdot (1-e^{-sT}) \cdot \frac{a}{s(s+a)}$$

$$\theta_o = \frac{aT \, e^{-sT}}{(1-e^{-sT}) \, s(s+a)}$$

The sampled output has the transform

$$\Theta_o^* = (\frac{1}{s}2)^* (1 - e^{-sT}) \left[\frac{s}{s(s+a)} \right]^*$$

$$= \frac{Te^{-sT}}{(1-e^{-sT})^2} \cdot (1-e^{-sT}) \cdot \frac{(1-e^{-sT}) e^{-sT}}{(1-e^{-aT}) (1-e^{-(s+a)T})}$$

$$= T(1-e^{-aT}) \frac{(e^{-sT})^2}{(1-e^{-sT})^2 (1-e^{-aT}e^{-sT})}$$

(25)

The character of the output is indicated by plotting the poles and zeros of Θ_o in the s-plane (see Fig. 13a). The transient frequencies of the sampled response (first order system response) are indicated by the pole-zero diagram of Θ_o^* in the e^{-sT} plane. (See Fig. 13b).
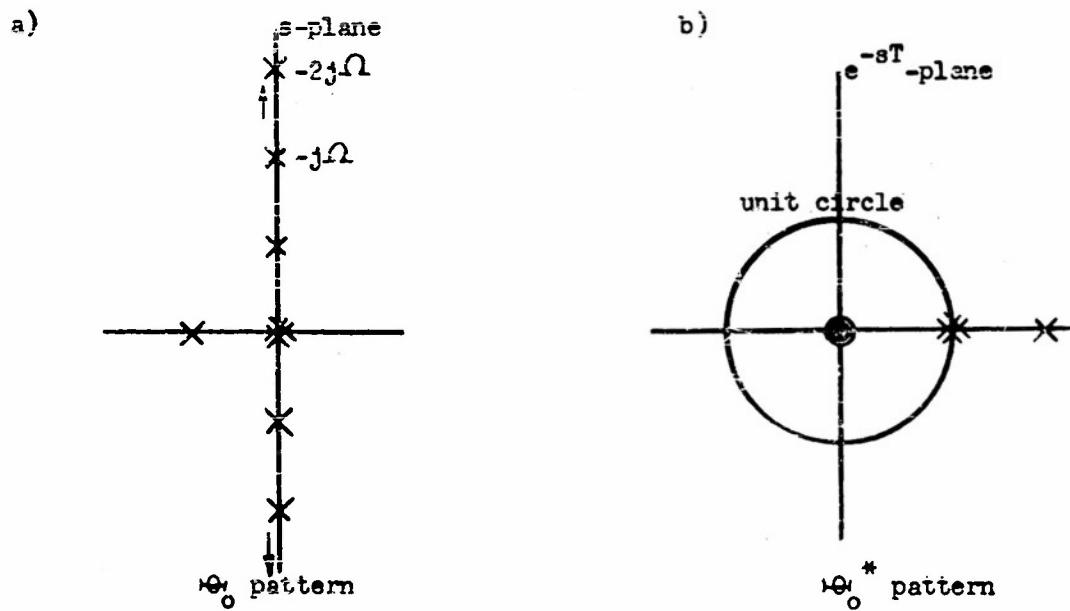

Fig. 13 Pole-zero pattern for Θ_o and Θ_o^*.

The s-plane diagram indicates that the overall response has the following components: a linear ramp component indicated by the double pole at the origin; possibly a step component (whose simple pole is obscured by the double pole); a decaying exponential transient indicated by the pole on the real axis; and a constant steady-state ripple, indicated by the sequence of simple poles on the imaginary axis. Transient ripple is absent.

The e^{-sT} plane diagram shows similarly that the sampled response indicates a linear ramp (double pole at $e^{-sT} = 1$), possibly a step, and an exponential transient (simple pole at $e^{-sT} = e^{-aT}$). We can find these components by breaking up the Θ_o^* expression into a partial fraction expansion which exhibits these modes. In order to see what to look for, we have recorded in Fig. 14 some simple modes and their transforms.

| Samples and their Envelopes	Transform of Sampled Time Function
1) Single Sample	1
2) Unit Step	$\dfrac{1}{1-e^{-sT}}$
3) Unit Ramp (slope = 1)	$\dfrac{Te^{-sT}}{(1-e^{-sT})^2}$
4) Real Exponential	$\dfrac{1}{1-e^{-\frac{T}{T}}e^{-sT}}$
5) Cosine	$\dfrac{1}{1+e^{-sT}}$
6) Ramp Cosine	$\dfrac{Te^{-sT}}{(1+e^{-sT})^2}$
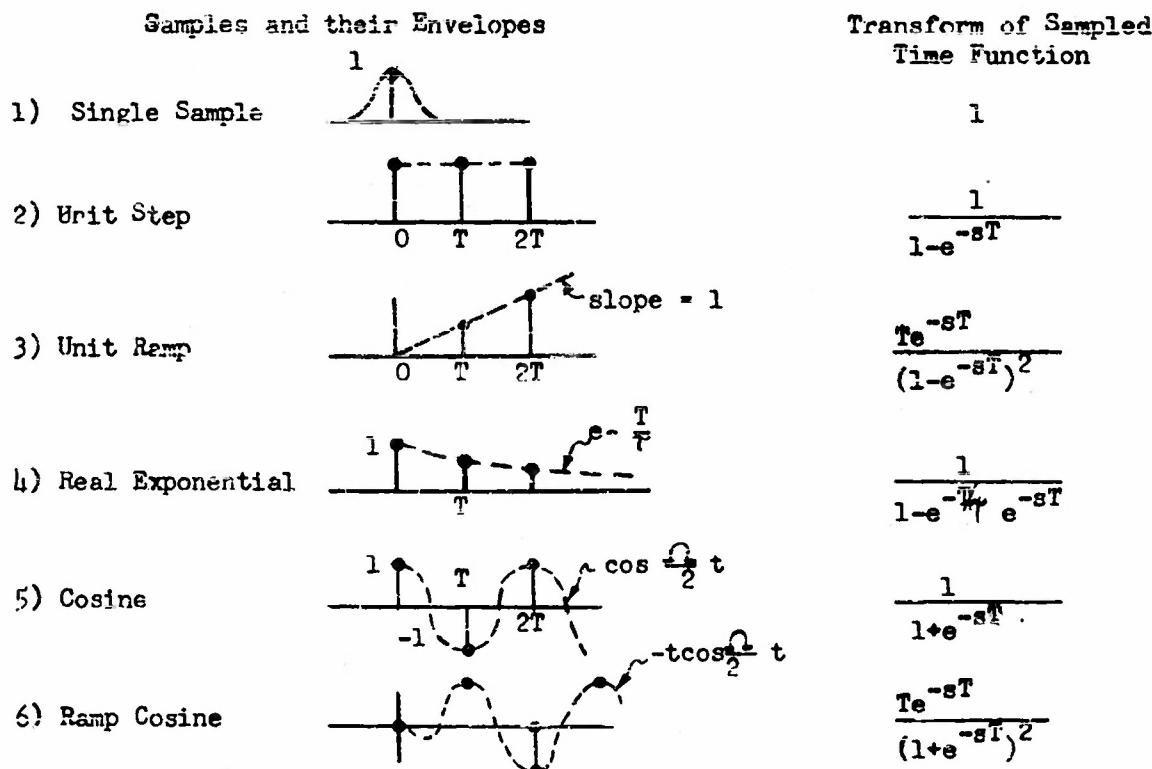
Fig. 14   Time functions and their transforms.

The Θ_o^* we have found breaks up into appropriate fractions in the following manner

$$\Theta_o^* = T(1-e^{-aT}) \frac{(e^{-sT})^2}{(1-e^{-sT})^2 (1-e^{-aT}e^{-sT})}$$

$$= T(1-e^{-aT}) \frac{\frac{1}{1-e^{-aT}}e^{-sT}}{(1-e^{-sT})^2} - \left(\frac{\frac{1}{1-e^{-aT}}}{1-e^{-sT}}\right)^2 + \left(\frac{\frac{1}{1-e^{-aT}}}{1-e^{-aT}e^{-sT}}\right)^2$$

$$= \frac{Te^{-sT}}{(1-e^{-sT})^2} - \frac{T}{1-e^{-aT}} \frac{1}{1-e^{-sT}} + \frac{T}{1-e^{-aT}} \frac{1}{1-e^{-aT}e^{-sT}} \qquad (26)$$

Note that contrary to the usual practice in such expansions we look for a term of the form $\frac{e^{-sT}}{(1-e^{-sT})^2}$ rather than $\frac{1}{(1-e^{-sT})^2}$, since the former can immediately be identified as the transform of a sampled ramp function. This change does not occasion any extra computational difficulty. Now we identify the first term of the result is a sampled unit linear ramp, the second term as a step of amplitude $= \frac{T}{1-e^{-aT}}$, and the third is a decaying exponential of time-constant $\frac{1}{a}$ and initial amplitude $\frac{T}{1-e^{-aT}}$. For convenience, then, the sampled output may be considered as arising from the sampling of the continuous signal $\Theta_e(t) = t - \frac{T}{1-e^{-aT}}(1-e^{-aT})$, and this signal may be taken as an approximation of the actual response $\Theta_3(t)$ where ripple effects have been neglected.

A sketch of the input to this system $\Theta_i(t)$, the actual response $\Theta_o(t)$, the sampled response $\Theta_o^*(t)$, and the envelope of the sampled response $\Theta_e(t)$ appear in Fig. 15.
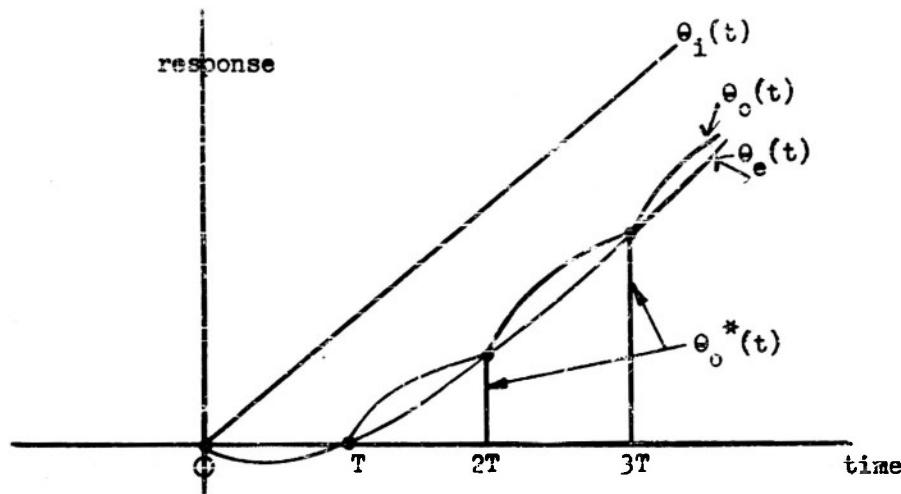
Fig. 15 Plot of approximations to the response.

The envelope curve of the samples approximates quite well the basic first

order behavior of the system. We are left with the problem of estimating

the ripple - of finding out what happens between samples. A glance at the

response curves of Fig. 15 shows that if we were to sample the output every

T/2 seconds instead of only every T seconds, the samples we would get would

show a component of oscillation of frequency Ω. The amplitude would indicate

very well the amount of ripple. As long as the system under consideration

has such a simple intersample behavior, this ripple estimation will be

reasonably accurate. In most control systems, since the controlled output

 member has low pass filter characteristics, the method may be satisfactorily

applied.

Suppose we wish to double-frequency sample the output of some

$K_a \dfrac{MDK}{} {}_b$ system. A second impulse modulator operating at one-half the period

of the first one is attached to the output and is synchronized with the

first modulator (see Fig. 16).

period T   A   period T/2

a)

b)

$$\Theta_0^{**} = (\Theta_1 K_a)^* DK_b^{**}$$
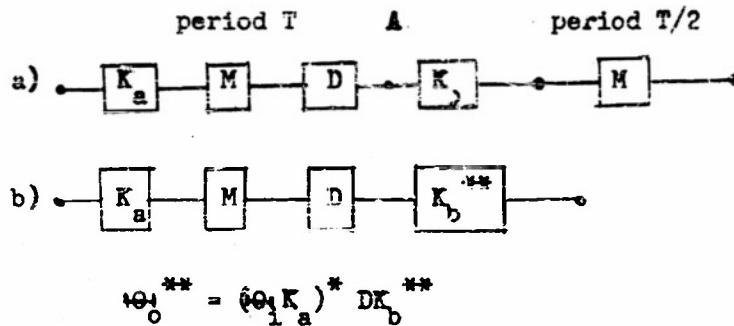
Fig. 16 Block diagram of system for ripple calculation.

Now at point A of Fig. 16a the signals are sampled, one sample arriving every T seconds. Nothing prevents us, however, from considering these samples to be arriving one every T/2 seconds. We just say that every other sample has a value of zero. Then at point A and the sampled output, the signals are sampled with period T/2 and an equivalent discrete signal filter exists given by

$$K_b^{**} = \frac{1}{T/2} \sum_{n=-\infty}^{+\infty} K_b(s+jn2\Omega) \qquad (27)$$

K_b^{**} can in most practical instants be found directly from the K_b^* used in determining the first order response. K_b^* is a rational function of e^{-sT}. K_b^{**} similarly is a rational function of $e^{-sT/2}$. K_b^{**} may be found from K_b^* merely by changing every T to T/2, provided that the correct square root is obtained. For instance, if somewhere in the expression for K_b^* some e^{-sT} was obtained actually by reducing the more general expression $e^{-(s+a)T}$ where the system constant, a, just happened to have the value $\frac{2\pi j}{T}$, then just replacing T by T/2 would give the result $e^{-sT/2}$ in the case of the reduced expression and $e^{-(s+a)T/2}$ in the second. But the former result is not now correct, since $e^{-aT/2} = e^{-\pi j}$, which is not equal to 1. Thus the simple replacement of T by T/2 may be performed only

if such reductions have not been made either because the original expression was retained or because the original expression could not be reduced at all.

Of course, we may always return to the full expression so as to be sure what to do, but time might be saved if we could tell at a glance, given K_b^*, whether K_b^{**} could be directly obtained from it. This simple substitution procedure is possible if the $e^{-(s+a_i)T/2}$ expressions resulting from the presence of modes of oscillation at the frequencies $s = -a_i$ cannot be simplified. The expressions cannot be simplified if these frequencies (indicated by poles of K_b in the s-plane) lie in the strip from $s = -j\Omega$ to $s = +j\Omega$. If any poles lie on the edges of this strip or outside it, the corresponding expression for K_b^* contains factors of the type $e^{-2\pi j}$, which if removed throw away essential phase information needed if $T/2$ is substituted for T.

As an example, consider two cases. In the first $K_b^{(1)} = \dfrac{1}{s^2 + \Omega^2}$, in the second $K_b^{(2)} = \dfrac{1}{s+a}$ (a is real). The pole-zero diagrams of these K_b are shown in Fig. 17d. We compute $K_b^{(1)*}$, $K_b^{(2)*}$, $K_b^{(1)**}$, $K_b^{(2)**}$.
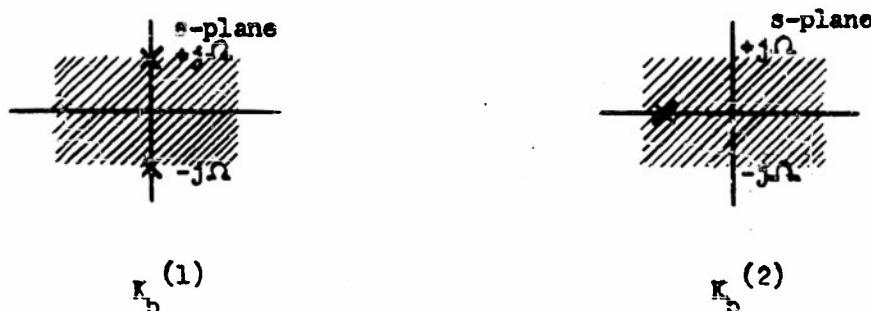
Fig. 17 Pole-zero diagrams for K_b.

$$K_b^{(1)*} = \left[\frac{1}{s^2 + \Omega^2}\right]^*$$

$$= \left[\frac{-\frac{1}{2j\Omega}}{s+j\Omega} + \frac{\frac{1}{2j\Omega}}{s-j\Omega}\right]^*$$

$$= \frac{1}{2j\Omega}\left[-\frac{1}{1-e^{-(s+j\Omega)T}} + \frac{1}{1-e^{-(s-j\Omega)T}}\right] \qquad (28)$$

$$= \frac{1}{2j\Omega}\left[-\frac{1}{1-e^{-sT}} + \frac{1}{1-e^{-sT}}\right]$$

$$= 0$$

$$K_b^{(2)*} = \left[\frac{1}{s+a}\right]^* \qquad (29)$$

$$= \frac{1}{1-e^{-(s+a)T}}$$

$$\bar{K}_b^{(1)*} = \frac{1}{2j\Omega}\left[-\frac{1}{1-e^{-(s+j\Omega)T/2}} + \frac{1}{1-e^{-(s-j\Omega)T/2}}\right]$$

$$= \frac{1}{2j\Omega}\left[\frac{-1}{1-e^{-\frac{j\pi}{2}}e^{-\frac{j\pi}{2}}} + \frac{1}{1-e^{+\frac{j\pi}{2}}e^{-\frac{sT}{2}}}\right] \qquad (30)$$

$$= \frac{1}{2j\Omega}\frac{2je^{-\frac{sT}{2}}}{\left(1+je^{-\frac{sT}{2}}\right)\left(1-je^{-\frac{sT}{2}}\right)}$$

$$= \frac{1}{\Omega}\frac{e^{-\frac{sT}{2}}}{1+e^{-sT}}$$

$$K_b^{(2)**} = \frac{1}{1-e^{-(s+a)T/2}} \qquad (31)$$

Note the difference in results. $K_b^{(1)**}$ is an entirely different expression that the final result for $K_b^{(1)**}$, while $K_b^{(2)**}$ may be obtained from $K_b^{(2)*}$ by just changing T to T/2 without going back to an earlier stage of the computation. There is in fact in the latter instance no earlier stage, since no reduction of the original summation is possible. Note how this behavior corresponds with the pole positions of the K_b in the s-plane.

Once K_b^{**} has been determined and the T/2 period sampled output transform $\theta_o^{**} = (\theta_i K_a)^* DK_b^{**}$ has been found, the poles and zeros of θ_o^{**} may be located in the $e^{-sT/2}$ plane. (Note that use has been made so far of the s, e^{-sT}-plane diagrams. The $e^{-sT/2}$-plane is a new one.) If K_b^{**} can be obtained from K_b^* by simple replacement of T by T/2, the poles and zeros of K_b^{**} in the $e^{-sT/2}$-plane are the same as the poles and zeros of K_b^* in the e^{-sT}-plane. These latter have already been found during the determination of the first-order response. The poles and zeros of $(\theta_i K_a)^* D$ in the $e^{-sT/2}$-plane may always be found by taking both square roots of each pole and zero of $(\theta_i K_a)^* D$ in the e^{-sT}-plane. For example, if $(\theta_i K_a)^* D$ has a zero at $9 \angle 60°$ in the e^{-sT}-plane, then in the $e^{-sT/2}$-plane there are corresponding zeros at $3 \angle 30°$ and $3 \angle 210°$. Therefore the pole and zero pattern provided by the $(\theta_i K_a)^* D$ part of θ_o^{**} in the $e^{-sT/2}$-plane is symmetric about both real and imaginary axes, since the pattern in the e^{-sT}-plane is already symmetric about the real axis.

There will arise in the above manner, a finite number of poles in the $e^{-sT/2}$-plane, each one corresponding to a component of the double-sampled output. We can, if we like, regard this pattern as indicating a higher approximation to the actual (unsampled) output response. However, the first order behavior has already been obtained through the e^{-sT}-plane diagram, and if the ripple is small, then this higher approximation will not lead to a significant change in the magnitude of the first-order response

components. Hence the $e^{-sT/2}$ plane may, if desired, be used solely for the purpose of estimating the ripple. This ripple magnitude is associated with the residues of poles in the $e^{-sT/2}$ plane not already identified with first-order response components. Such residues may be found analytically or by graphical measurement.

The above discussion is now illustrated by finding the ripple in the system of Fig. 9, whose first order response has been found and sketched in Fig. 12. First note the $K_b = \dfrac{a}{s(s+a)}$ has poles within the strip $s = +j\Omega$ to $s = -j\Omega$. Therefore K_b^{**} is found directly from K_b^{*}, and the pattern of K_b^{**} in the $e^{-sT/2}$ plane is the same as that of K_b^{*} in the e^{-sT} plane. Θ_o^{**} may be found immediately.

$$K_b = \frac{a}{s(s+a)} \qquad K_b^{*} = \frac{e^{-sT}(1-e^{-aT})}{(1-e^{-sT})(1-e^{-aT}e^{-sT})} \qquad K_b^{**} = \frac{e^{-\frac{sT}{2}}(1-e^{-\frac{aT}{2}})}{\left(1-e^{-\frac{sT}{2}}\right)\left(1-e^{-\frac{aT}{2}}e^{-\frac{sT}{2}}\right)} \tag{32}$$

$$\Theta_o^{**} = (\Theta_i K_a)^{*}D K_b^{**} = \frac{Te^{-sT}}{1-e^{-sT}} \cdot \frac{e^{-\frac{sT}{2}}\left(1-e^{-\frac{aT}{2}}\right)}{\left(1-e^{-\frac{sT}{2}}\right)\left(1-e^{-\frac{aT}{2}}e^{-\frac{sT}{2}}\right)}$$

$$= T(1-e^{-\frac{aT}{2}}) \frac{\left(e^{-\frac{sT}{2}}\right)^3}{\left(1-e^{-\frac{sT}{2}}\right)^2\left(1+e^{-\frac{sT}{2}}\right)\left(1-e^{-\frac{aT}{2}}e^{-\frac{sT}{2}}\right)} \tag{33}$$

A sketch of the Θ_o^{**} pattern in the $e^{-\frac{sT}{2}}$ plane is shown in Fig. 18.
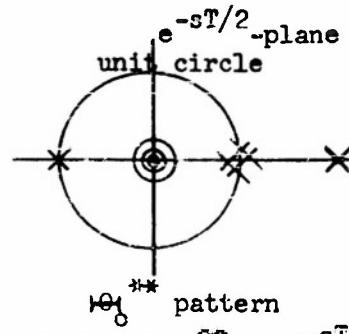
Fig. 18. Pole-zero pattern of Θ_o^{**} in $e^{-sT/2}$-plane.

There are four possible output components indicated by the poles in Fig. 18. Compare this pattern with that of θ_o^* in the e^{-sT} plane in Fig. 13b. The poles at $e^{-sT/2} = 1$ and $e^{-sT/2} = e^{-aT/2}$ in Fig. 18 correspond to those in Fig. 13b for which $e^{-sT} = 1$ and $e^{-sT} = e^{-aT}$ and indicate as before a ramp, a step, and a decaying exponential component in the output. However, there is now an extra pole at $e^{-sT/2} = -1$ in the θ_o^{**} pattern. This corresponds to a term $\dfrac{A}{1+e^{-sT/2}}$ in the partial fraction expansion of θ_o^{**} and represents (refer to Fig. 11) a sampled cosine oscillation of amplitude A and frequency Ω. We note from the s-plane diagram of θ_o (Fig. 10a) that there are no transient ripple components and are not surprised that we obtained only the one ripple pole. Evaluating A we get

$$A = (1+e^{\frac{-sT}{2}}) \cdot \theta_o^{**} \Big|_{e^{\frac{-sT}{2}} = -1}$$

$$= \frac{-T(1-e^{\frac{-aT}{2}})}{4(1+e^{\frac{-aT}{2}})}$$

(34)

so that the ripple may be considered as arising from the samples of a wave $-\dfrac{T}{4}\dfrac{(1-e^{-aT/2})}{(1+e^{-aT/2})}\cos\Omega t$, and this wave may be added to the first-order response approximation to obtain a continuous wave which fits very well to the actual output time function $\theta_o(t)$. (See Fig. 19.)
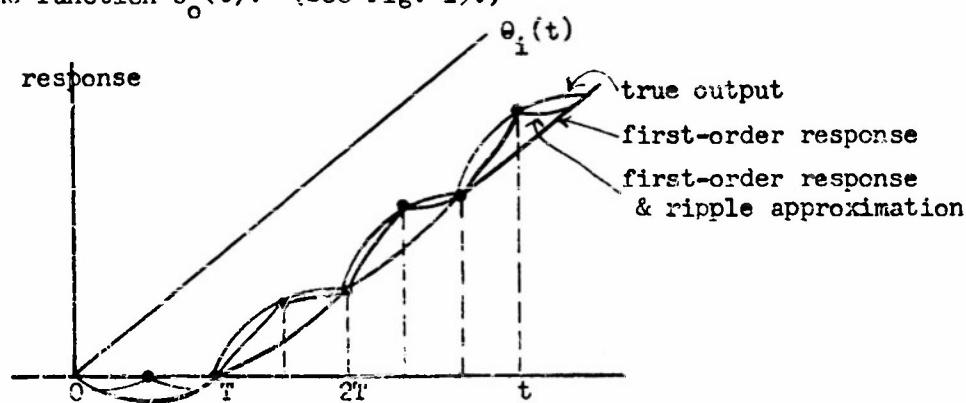

Fig. 19. Comparison of approximations to $\theta_o(t)$.

The step component of the first-order response should be modified, however, either by redetermining it from the Θ_0^{**} transform, or by merely matching the ripple to what is already known to be the value of the samples at $t = 0$, T, $2T$ ... . Thus to the first order wave one should really add $\frac{T}{4} \frac{(1-e^{-aT/2})}{(1+e^{-aT/2})} (1 - \cos \Omega t)$ to insure that at $t = 0$, T, $2T$ ... the samples of the new curve have the same value as before. The second-order approximation includes the ripple effect. The envelope curve of the double frequency samples of the output is then:

$$\Theta_e(t) = t - \frac{T}{1-e^{-aT}} (1-e^{-aT}) + \frac{T}{4} \frac{\left(1-e^{\frac{-aT}{2}}\right)}{\left(1+e^{\frac{-aT}{2}}\right)} (1 - \cos \Omega t) \qquad (35)$$

From this expression we can find how the ratio of the step lag in response to the ripple amplitude behaves with a change in $\frac{1}{a}$, the system time-constant, as well as noting their absolute values. This method of approximation to the response of the sampled-data system, then, readily provides enough information for design. It offers means for determining just how system parameters should be set for optimum response characteristics.

Summary

The preceding section discussed the analysis of the response of a basic cascade sampled-data system. It was mentioned that any sampled-data system of arbitrary complexity could be reduced to an equivalent system containing at most a number of parallel branches of the standard cascaded sequence of elements, continuous signal filter, impulse modulator, discrete signal filter, and second continuous signal filter. Therefore, the analysis of the response of the basic $K_a MDK_b$ sequence is sufficient to allow analysis of any system. The response of the $K_a MDK_b$ system was then calculated exactly. Finally, methods of approximation to this response were given which had a

practical value in being easily applied to obtain an overall picture of the
main features of the response and their dependence on system parameters.
It remains to demonstrate in Sec. 2.2 how the general sampled-data system
is reduced to a parallel set of standard $K_a MDK_b$ branches.

2.2 Reduction of the General Sampled Data System

This section is devoted to a method by which an equivalent network
is obtained for a general system whose response is easy to compute by
straightforward application of the methods of the preceding sections.

2.21 Symbolism

To facilitate discussion of the points to follow it is necessary
to introduce a shorthand for dealing with sampled-data systems. The reader is
probably familiar with the way in which the use of transfer functions in
the transformation of conventional linear networks into various circuit
equivalents leads to a compactness of notation and an appreciation of the
circuit behavior which cannot be equalled by the use of the circuit differential
equations themselves.

Unfortunately sampled-data systems cannot be entirely described
by such transfer functions. It has been noted in previous sections that
the discrete and continuous filters may be so described but that the sampling
process, being essentially a modulation, cannot be represented by a simple
transfer function (at least not by one of the ordinary time-invariant type).
One easy way to appreciate this fact is to note that the wave shape of the
output of a sampler (impulse modulator) depends not only on the wave shape
of the input, as it would if it were describable by an ordinary transfer
function, but in addition depends on the relative timing of the input and
sampling.

However, the sampling process is linear, the output for a sum of
inputs being equal to the sum of the responses from the inputs applied one at
a time. This superposition property is an essential feature of the methods
to follow. Because of this linearity of the separate components of a sampled-data
system (i.e., impulse modulator, digital filter, continuous filter), the system
itself is linear in its entirety. The store of superposition methods,
suitably extended, is therefore available for the analysis of sampled-data
systems.

Since a sampled-data system is linear but not entirely describable
by transfer functions, the more general concept of linear erations may be
introduced. Every component of a sampled-data system operates linearly on
its input to produce its output. Such an operation may be indicated by a
juxtaposition of input symbol and operational symbol as indicated in Fig. 20.

Linear
component

Θ_i L Θ_o

$$\Theta_o = \Theta_i L$$
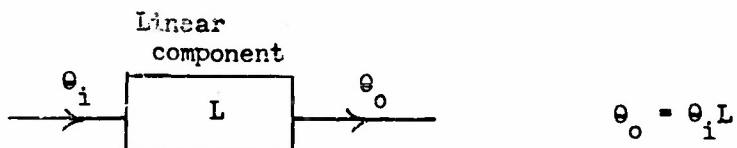
Fig. 20. Representation of a general linear component.
Note that the order in which the symbols appear is significant. $\Theta_i L$ means
that L operates on Θ_i. This order is chosen arbitrarily to provide a certain
isomorphism of the symbolism with the way in which electrical engineers
usually draw diagrams -- i.e., signal flow from left to right. Thus, consider
the cascade of linear components in Fig. 21 and the corresponding operational
expression.

$$\Theta_A = \Theta_i L_1$$

$$\Theta_B = \Theta_A L_2$$

$$\Theta_o = \Theta_B L_3$$

$$\Theta_o = (\Theta_A L_2) L_3 = \left[(\Theta_i L_1) L_2\right] L_3$$

$$\underline{or} \quad \Theta_o = \Theta_i L_1 L_2 L_3 = L$$

$$\text{where } L = L_1 L_2 L_3$$
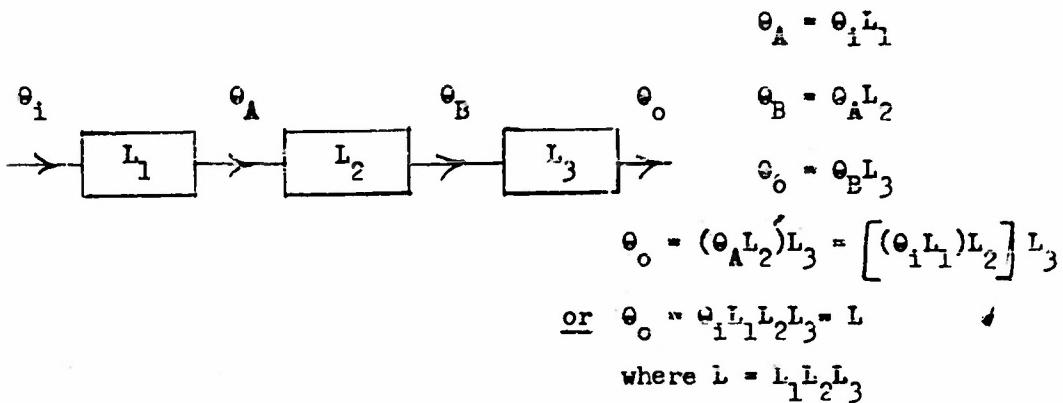
Fig. 21. Several linear elements in cascade.

The overall operation of a cascade of separate operations is indicated
by the symbols $L_1 L_2 L_3$ in the order in which the separate operations are
performed. The fact that the operations are linear does not imply that they
are commutable, that is, that $L_1 L_2 = L_2 L_1$. It is known, however, that if
L_1 and L_2 refer to operations performed by continuous filters they are
commutable, the order of cascading ordinary constant coefficient linear
networks being immaterial. The impulse modulating operation, however,
obviously does not commute with a continuous filtering operation, since
the output must be in the form of samples in the first instance but in the
form of a continuous wave in the second. It must also be noted that the
filter operations also cannot be commuted if there is a sampling operation
between them.

Because of the linearity of the systems under consideration, the
operation performed by a paralleled set of components may be indicated by
the sum of the separate operations. Refer to Fig. 22.
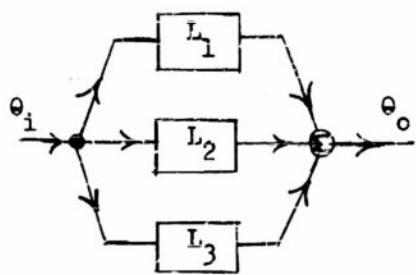
$$\Theta_o = \Theta_i L_1 + \Theta_i L_2 + \Theta_i L_3$$

$$= \Theta_i \left[L_1 + L_2 + L_3 \right]$$

$$= \Theta_i L$$

where $L = L_1 + L_2 + L_3$

Fig. 22. Linear elements in parallel.

We now allocate a specific symbol to the operations performed in a sampled-data system:

M, operation of sampling or impulse modulation;

K, operation of continuous filtering;

D, operation of discrete filtering;

T, overall operation (transmission).

Because of the sampled signals which are involved, the digital filtering D is always preceded by either another digital filtering D' or impulse modulation M.

An important restriction must be made on the systems to be considered. It will be assumed that all impulse modulators are running at the same frequency and in the same phase (or at least very nearly in the same phase). Furthermore, it will be assumed that no questions arise as to whether various impulse modulators take their samples just before or just after input-wave discontinuities produced by previous sampling operations. The restriction on frequency and phase is essential; the restriction on the sampling of discontinuities is not essential and is introduced only for convenience in the following discussion. The neglected effects are easily taken into account; how this can be done is best described later after the basic ideas have been presented. Because of the above stipulations we can make indiscriminant use of the symbol M for every sampling operation in the sampled-data system.

Very often a cascade operation occurs in which more than one sampling appears; see Fig. 23.

sample signals

$T = MDKM$

$= M(DK)^*$

$= MDK^*$

overall operation, T ⟶
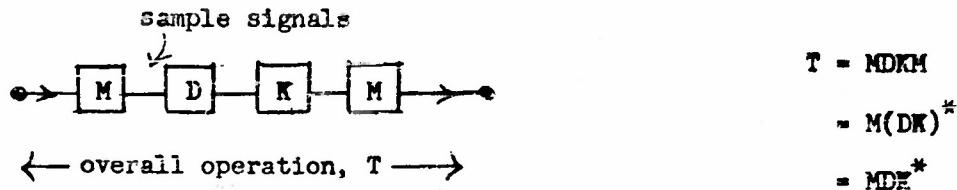
Fig. 23. Reduction removing all but the first modulator.

In this case as discussed in Sec. 2, the system may be simplified to an equivalent system of a modulator followed by some discrete filter given by

$$Deq(s) = \frac{1}{T} \sum_{n=-\infty}^{+\infty} D(s+jn\Omega)K(s+jn\Omega)$$

$$= \frac{1}{T} \cdot \overline{D(s)} \sum_{n=-\infty}^{+\infty} K(s+jn\Omega).$$

(36)

$Deq(s) = D(s)K^*(s)$, where the * refers to the indicated summation. Use has been made of the fact that $D(s)$ is periodic.

We use the operational symbols in the same way as the transfer function symbols above. That is

$$Deq = (DK)^* = DK^*$$

(37)

so that

$$MDKM = MDK^*$$

(38)

2.22 Reduction of Series-Parallel Sampled-Data Network.

The operational symbolism introduced above may be used in the finding of equivalent circuits. This simplest application of these methods is the reduction of a series-parallel system to simpler form.

From the linearity of a sampled-data system it follows that the distributive law of "multiplication" holds with respect to the operations if correct order of the factors is maintained. Refer to Fig. 24.
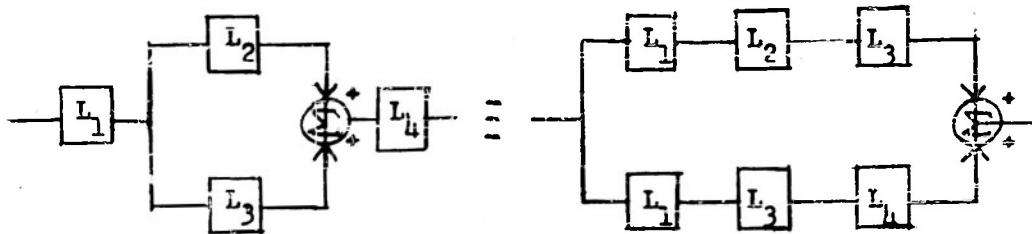
$$T = L_1 \ (L_2+L_3)L_4 \ = \ L_1L_2L_4 + L_1L_3L_4$$

Fig. 24. Reduction of a series-parallel network.

A this point it is well to take stock of the relationships involved in operator manipulations. Table I gives a summary.

TABLE I

Summary of Operator Manipulations

Conventions in Notation

L = any sampled-data system operation

M = impulse modulation, sampling

K = continuous signal filtering

D = discrete signal filtering

K^* = the derived discrete filter equivalent operation for the operation K followed by M when the input signals are discrete.

T = the symbol for some overall combination of operations which relates an input signal to an output.

A = a simple gain or attenuation

$L_1 L_2 L_3$ = operations L_1, L_2, L_3 performed successively in the order 1,2,3.

$L_1 + L_2 + L_3$ = operations performed separately, results added.

Symbolic Relations

a) Commutability

$$MK \neq KM \qquad\qquad K_1 K_2 = K_2 K_1$$

$$DK \neq KD \qquad\qquad D_1 D_2 = D_2 D_1$$

MD $\neq$ DM, unless the input signals are discrete.

All symbols commute with the gain operator, A.

b) Combination, factoring

$$L_1(L_2 + L_3) = L_1 L_2 + L_1 L_3$$

$$(L_2 + L_3)L_1 = L_2 L_1 + L_3 L_1$$

$$(L_1 + L_2)(L_3 + L_4) = L_1 L_3 + L_1 L_4 + L_2 L_3 + L_2 L_4$$

c) Cascade Equivalents

$$MDKM = MDK^* \qquad\qquad (DK)^* = DK^*$$

$$(K_1 K_2)^* \neq K_1^* K_2^*$$

Through use of these symbols, any series-parallel sampled-data system operation may be written in symbolic form. Reduction to a simpler equivalent operation, if such exists, may be effected by using the relations mentioned above. Fig. 25 gives an example of such a reduction. For simplicity the blocks of the block diagram are omitted. Summing points are indicated by nodes, the output on <u>each</u> of the lines from a node being the sum of all the inputs to that node. Operations are written above corresponding branches between nodes.
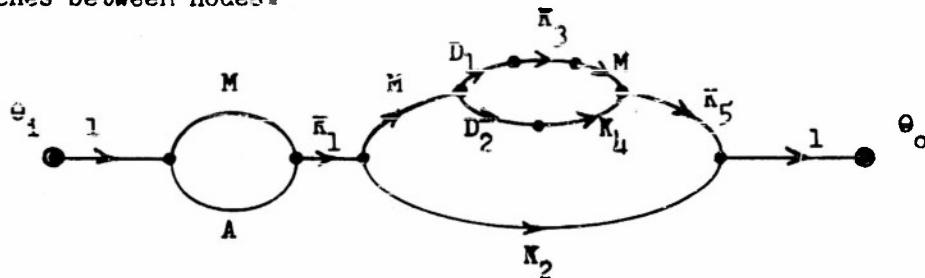
(a) Original system

$$T = (M+A)\overline{K}_1\left[\overline{K}_2 + M(\overline{D}_1\overline{K}_3 M + D_2\overline{K}_4)\overline{K}_5\right]$$

$$= (M+A)\left[\overline{K}_1 K_2 + \overline{K}_1 MD_1 \overline{K}_3 M\overline{K}_5 + \overline{K}_1 MK_2 K_4 K_5\right]$$

$$= (M+A)\left[\overline{K}_1 K_2 + \overline{K}_1 MD_1 K_3^* \overline{K}_5 + \overline{K}_1 MD_2 \overline{K}_4 K_5\right]$$

$$= M(\overline{K}_1 \overline{K}_2 + K_1^* D_1 K_3^* \overline{K}_5 + K_1^* D_2 K_4 K_5)$$

$$+ A\overline{K}_1 K_2 + AK_1 MD_1 K_3^* \overline{K}_5 + AK_1 MD_2 \overline{K}_4 K_5$$

$$\text{or} = M\overline{K}_1 K_2 + MK_1^* D_1 K_3^* \overline{K}_5 + MK_1^* D_2 K_4 K_5$$

$$+ AK_1 K_2 + AK_1 MD_1 K_3^* \overline{K}_5 + AK_1 MD_2 \overline{K}_4 K_5$$
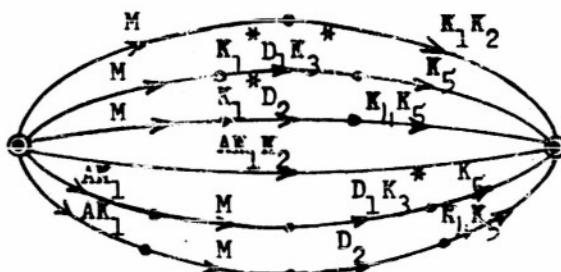
(b) Equivalent system in standard form.

Fig. 25. Reduction of a complicated system to standard form.

The results of this reduction may not at first glance appear simpler than the original expression. In terms of economy of symbols this observation is correct. However, the object of practical interest is to be able to compute the output from such a system when the input is given. The new form permits this computation; the old form does not, unless one simultaneously during the computation goes through the various steps mentally by which the equivalent was obtained.

Notice from the example that a series-parallel network can always be reduced to a parallel set of operations of the form $K_a MDK_b$. (The operation K is a special case of this form, when M and D are absent.) To compute the overall output, it is only necessary to sum each of the outputs from each $K_a MDK_b$ operation

The computation of the response to the $K_a MDK_b$ operation has already been described in Sec. 2.11, where it was shown that, given the transfer functions $K_1(s)$, $K_2(s)$, $D(s)$ and the signal transform $\Theta_i(s)$, the output is,

$$\Theta_o(s) = \left[\frac{1}{T} \sum_{n=-\infty}^{+\infty} \Theta_i(s+jn\Omega) K_1(s+jn\Omega) \right] D(s)K_2(s)$$

$$= \left[\Theta_i(s)K_1(s) \right]^* D(s)K_2(s). \tag{39}$$

Every term in the reduced series-parallel equivalent can be handled in this way. Note that if two or more terms of the reduced equivalent have the same operations before (or after) the sampling, a factoring is possible which still preserves the simplicity. For instance suppose

$$\begin{aligned}T &= K_1 M D_1 K_2 + K_1 M D_2 K_3 \\ &= K_1 M (D_1 K_2 + D_2 K_3),\end{aligned} \tag{40}$$

so that the output may be computed as follows,

$$C_0^*(s) = \left[\Theta_1^*(s)K_1^*(s)\right]^* \left[D_1^*(s)K_2^*(s) + D_2^*(s)K_3^*(s)\right] \tag{41}$$

which is seen to be the same result that would have been obtained from finding the responses separately.

2.3 Reduction of Feedback Sampled-Data Networks

The reduction of series-parallel networks is very straightforward since the overall operation, T, may immediately be written by inspection and subsequently reduced. It is not so obvious, however, how to write down an expression for the overall operation performed by a feedback system, since there are constraint relationships among the various signals in the system. The purpose of this section is to show how any feedback system can be reduced to a series-parallel system whose output can be directly evaluated in the manner previously described.

The entire procedure hinges on two conditions - the linearity of the system in general, and the possibility in particular of finding a directly-connected circuit which performs the same operation as a feedback loop containing a series-parallel combination of sampled-data system components.

The remainder of the discussion makes use of flow-graph methods.
The translation from block-diagram to flow-graph language is shown in
Fig. 26.

Block Diagrams   Flow Graphs

tie points,
summation points   nodes

blocks   branches
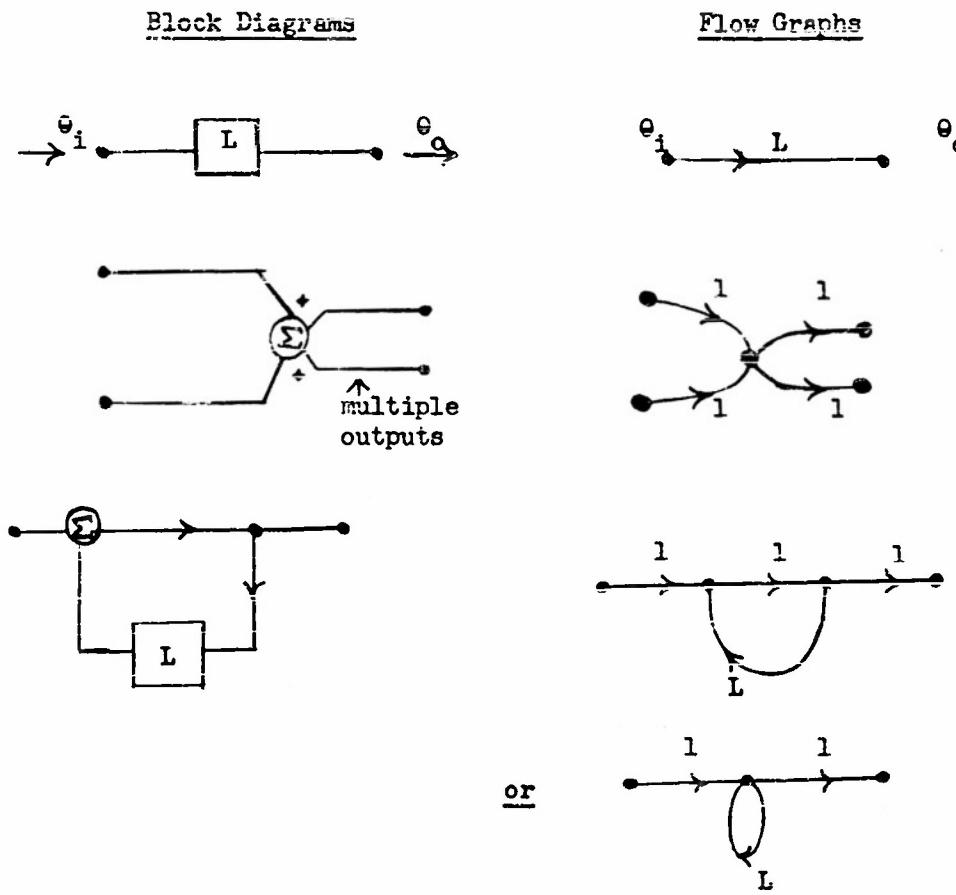
Fig. 26 Block diagrams and their corresponding flow graphs.

The flow graph is simply a convenient way to draw block diagrams.
Actually every operation which involves manipulation of a flow graph
could be applied equally well to a block diagram. The essential idea here
is that block-diagram or flow-graph techniques can be applied equally
well to servo systems, feedback amplifiers, linear computing systems,

or any other system comprised of linear operations. One can show
interdependences of signals with flow graphs for non-linear systems,
but the reduction manipulations do not apply because the reductions
are based on superposition.[7]

The problem in reducing a flow graph is to reduce the number
of nodes to a minimal number and to reduce the number of branches to a
minimal number. For a conventional linear system a fully reduced flow
graph is one branch connecting the input node and the output node.
Reductions are of three kinds: (1) A node between two cascaded elements
can be removed; (2) parallel branches between the same pair of nodes
can be replaced by a single branch; (3) reductions of self-loops and
their nodes can be made. When one has made as many reductions of
types 1 and 2 as possible, there will be a number of residual nodes
and two kinds of loops. Self-loops will be made of one branch and
will start and end on the same node, as in Fig. 28a. System loops will
involve several branches and several nodes. By removing self-loops
and their nodes, one can make successive reductions until the flow
graph is completely reduced and has no loops of either type. By this
procedure, all loops are reduced to self-loops and then removed. In
any specific problem, the reduction can proceed in several fashions,
but it is well to use the one which involves least work.

---

[7] Mason, S.J., Some Properties of Signal Flow Graphs (to be published
in IRE Proceedings).

Steps 1 and 2 can usually be done at once by picking the residual

nodes and then evaluating the transmissions between them by inspection.

For example, consider the feedback network of Fig. 27.
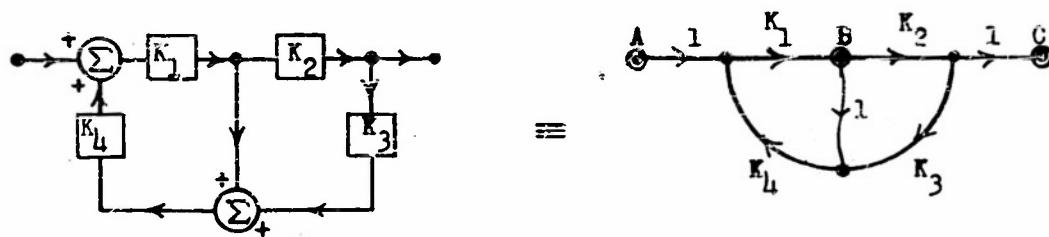
Fig. 27. A block diagram and its flow graph which will be reduced.

Because of linearity we can pick any nodes, say the circled ones A,B,C, and,

by writing the relationships imposed on the signals which leave one node

and arrive at another, the remaining nodes being left out of the picture

for that determination, can draw an equivalent circuit containing only

the circled nodes. In the example:

A to A	0	B to A	0	C to A	0
A to B	K_1	B to B	$(K_2 K_3 + 1)K_4 K_1$	C to B	0
A to C	0	B to C	K_2	C to C	0

so that the reduction to Fig. 28a may be made.

(a)   (b)
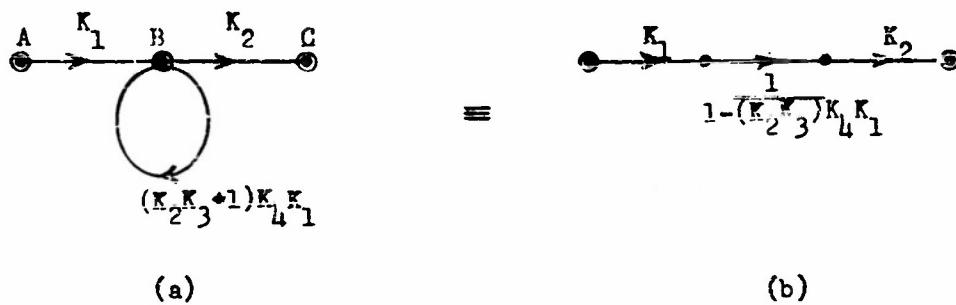
Fig. 28. The reduced graph for Fig. 27.

Since the operations indicated are continuous signal filtering, we may look
on the operation K as the transfer function K and apply a well-known result
that the loop of Fig. 28 is equivalent to a directly connected element whose
transfer function is $\dfrac{1}{1-(K_2 K_3 +1)K_4 K_1}$. The overall response of the network
is then given by $T = \dfrac{K_1 K_2}{1-(K_2 K_3 +1)K_4 K_1}$. See Fig. 28b.

Two essential features of this process should be noticed. One
is that the nodes to be retained - residual nodes - are chosen so that the
operations on signals flowing among these nodes can be written directly
in series-parallel form. In other words, the nodes are so chosen that their
deletion from the diagram interrupts all the feedback paths. Of course,
the input and output nodes are also retained. The other is that an equivalent
direct operation must be found for the self-loop feedback loops which arise
in the residual node graph. In conventional, continuous-signal systems,
this can always be done by manipulating transfer functions.

In sampled-data systems, the picture is slightly different.
In the example up to the point where the feedback loop was removed from
the residual graph there need have been no distinction between the treatment
of conventional and sampled-data systems. The various K's could just as
easily have been any sampled-data system operators, provided proper attention
was paid to the ordering of the symbols. However, since there is no conventional
transfer function which describes the general sampled-data system operator,
the method applied to remove the loop in the conventional system cannot
be used in sampled-data systems unless the loop operations are describable
by transfer functions; that is, if the loop transmission may be described
as a discrete signal filter or as a continuous-signal filter. In this

latter instance the reduction can proceed in the usual manner with the interpretation that the transfer function, $\frac{1}{1-T(s)}$, has as its counterpart an operator $T^1 = \frac{1}{1-T}$. Note, however, that when the loop contains a sampling operation without a sampled end signal (e.g., $T = KM$) there is no transfer function $\frac{1}{1-KM}$ and the operation $\frac{1}{1-KM}$ makes no sense in the above respect.

We are left, therefore, with the task of finding some direct operation with which to replace the loop operations occurring in the residual graph of a sampled-data system in the general case. We begin by considering the simple case shown in Fig. 29 for the loop operation $K_1 MDK_2$.
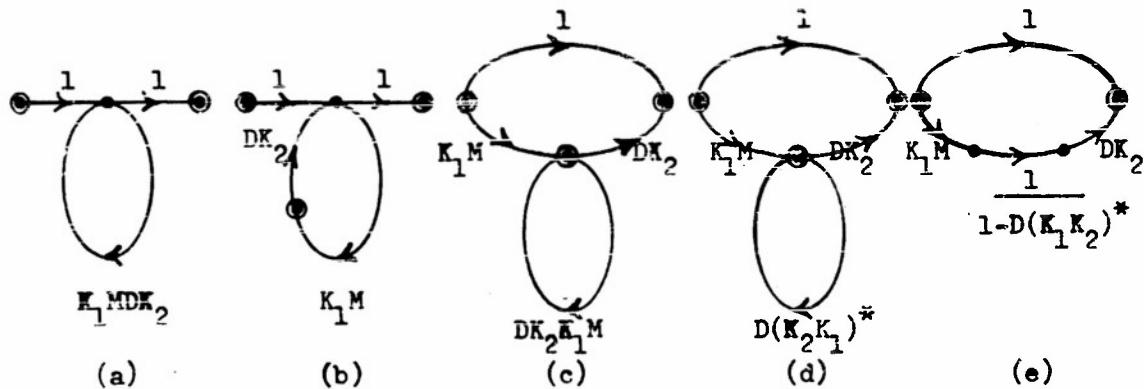
Fig. 29. Handling a self-loop with an impulse modulator.

Instead of accepting the form of the graph as shown in Fig. 29a we make the following modification. As residual nodes of a new graph we pick the input, the output, and the node just following the sampling operation in the original graph. This set is evidently satisfactory, since the feedback has been removed through this last choice. The new residual graph is shown in Fig. 29c.

Now the signals entering the residual loop node are sampled.
Also the last operation which appears in the loop is a sampling. Therefore,
an equivalent discrete filter may be found by a process previously described.

$$D\bar{K}_2\bar{K}_1M = D(\bar{K}_2\bar{K}_1)^*, \text{ for sampled inputs.} \qquad (42)$$

Now the loop operations may be described by transfer functions, and sense
may be given to the operation $\dfrac{1}{1-D(\bar{K}_1\bar{K}_2)^*}$. The network has thus been reduced

a series-parallel form which is without further transformation in the
standard form of a parallel set of $\bar{K}_a MD\bar{K}_b$ branches. The response is directly
calculable by the methods of the preceding section, once the input is prescribed.

An elaboration of the above procedure permits the reduction of any
feedback network to series-parallel form. The steps to be followed are:

1. Choose residual nodes in the following way. Use the input and
output nodes as residual nodes. Break up as many feedback paths as possible
by picking residual nodes at points in the graph at which the signal is
of a purely sampled nature. (The insertion of branches of unity transmission
may be necessary.) If this procedure does not result in the interrupting
of all feedback, it must be because there are some feedback paths left
along which all the components are continuous filters. Pick enough extra
residual nodes to interrupt this feedback. Note that there are three
classes of residual nodes: a) input-output, b) nodes at which the signals
are purely sampled, and c) nodes whose self-loop transmissions can be
described by a product of continuous-signal filter transfer functions.

2. Simplify the graph by inserting the equivalent straight
through transmission for the loop transmission of one of the nodes of
class c. Re-evaluate the transmissions among all remaining nodes. The
resulting graph has one less node of class c but has the original number of

class b nodes. Now repeat this reduction until the class c nodes are removed. Only class b and a nodes remain.

3. Simplify the graph further by inserting the equivalent straight-through transmission for the loop transmission of one of the class b nodes. This step is possible since the loop transmission is of the form $(K_1 MD_1 K_2 + K_3 MD_2 K_4 + \ldots)$ operating on purely sampled signals. Therefore, an equivalent loop-transmission transfer function exists given by $(K_1{}^* D_1 K_2 + K_3{}^* D_2 K_4 + \ldots)$, and the equivalent straight-through transfer function is $\dfrac{1}{1-(K_1{}^* D_1 K_2 + \ldots)}$. Now re-evaluate the transmissions among all remaining nodes. The resulting graph has one less node of class b. Repeat this process and remove all class b nodes. The only residual nodes left are the input and output; the feedback has been removed and the graph is now in series-parallel form. The canonic KMDK form can now be found and the response to an input computed.

Note that in the above process it is essential to start removing class c nodes first and then proceed to class b residual nodes. Otherwise, if a class b node is removed first, the loop transmissions in the reduced graph may be such that some of original class c nodes lose their character. However, if a class c node is removed the signals at class b nodes remain sampled and class b nodes remain as such in the reduced graph.

As an illustration of the above methods consider the sampled-data
feedback system of Fig. 30a.

a)
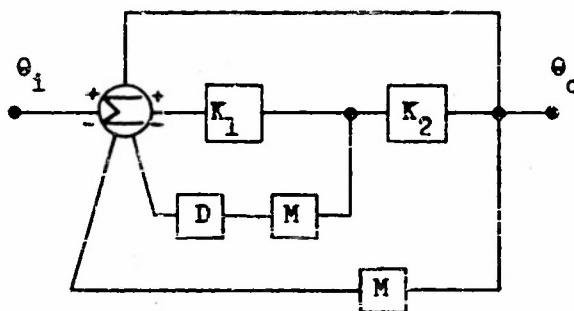
b)
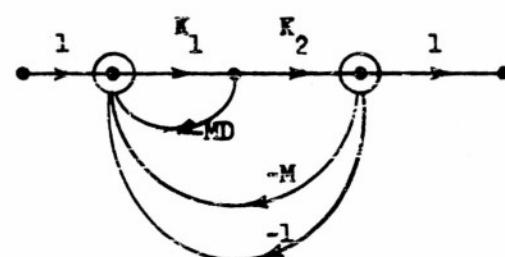
c)
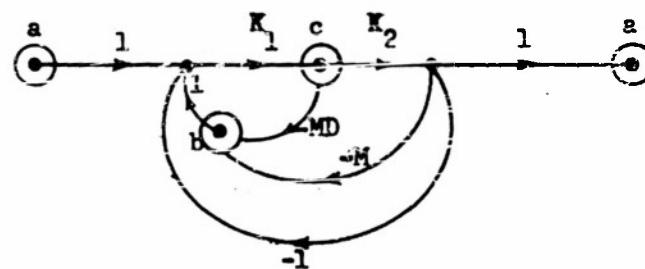
d)
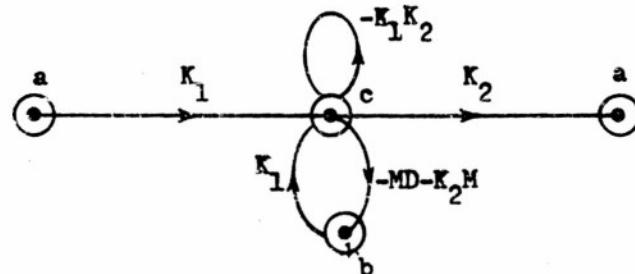
Fig. 30.   Reduction of a flow graph for a complicated sampled-data system.

(Continued on next page)
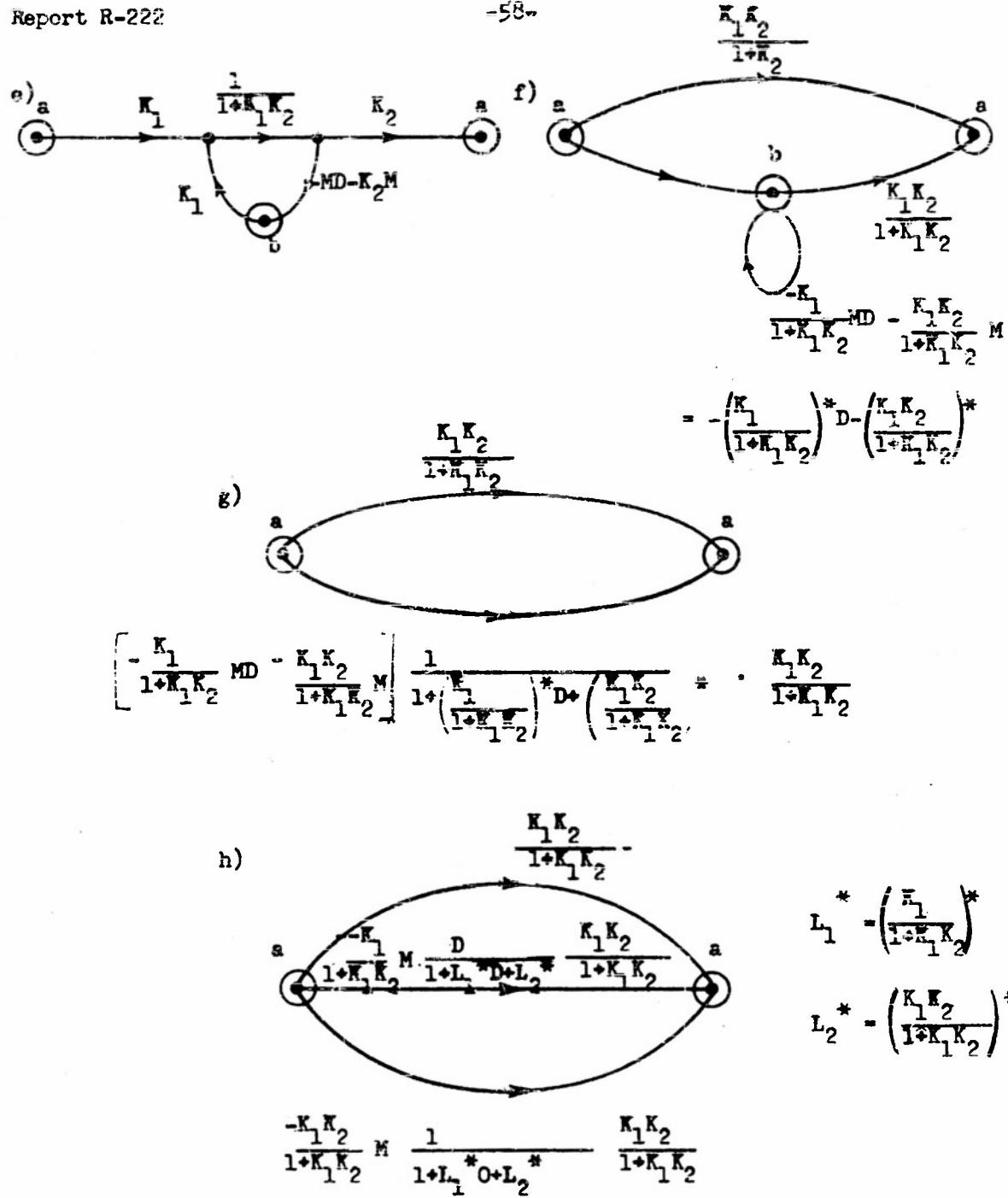
Fig. 30. Reduction of a flow graph for a complicated sampled-data system.

The system in flow graph form is shown in Fig. 30b. An extra branch of unity
transmission is inserted as in 30c so that a node b may be chosen which
interrupts feedback and to which the signals are purely sampled. There remains
a feedback path which cannot be broken by such a class b node. A class c
residual node is assigned which completes the break-up of feedback. In
Fig. 30c these residual nodes are circled and their class is noted. Fig. 30d
is the reduced graph showing only these nodes. The equivalent straight-through
connection is made for node c in Fig. 30e, and the graph is reduced to
that of 30f. The only class c node has been removed; now the class b
node is removed in the same way and the graph simplified in steps g,h.
The final result is a network in standard form which permits response
computations according to the methods of Section 2.22.

2.31 A Sampled-Data Servomechanism

One sampled-data feedback system of particular interest is the
simple servo system shown in Fig. 31.
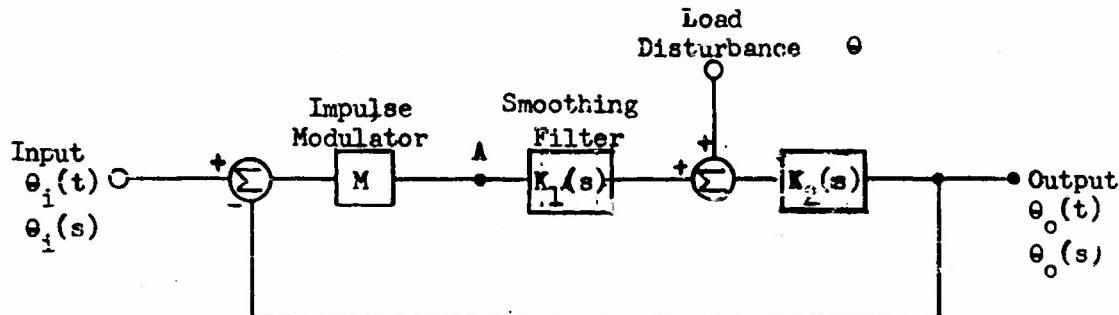
Fig. 31. Block diagram of a single-loop servo system of particular
interest.

In this diagram the impulse modulator is followed by a filter with transfer
function $K_1(s)$ which smooths out the discrete error signals from the modulator

and delivers this error to the controlled element described by the transfer function $K_2(s)$. Input, output, and load-disturbance time functions and their Laplace transforms are noted on the diagram.

Two response characteristics of this system are of interest - the output resulting from a given input and that resulting from a given load disturbance. In order to make possible the calculation of these responses we transform the given circuit into one without feedback. Fig. 32 shows the application of the general procedure for reduction given above culminating in networks of standard form.

(a) Equivalent System for
    Input-Output Response

(b) Equivalent System for Load
    Disturbance-Output Response
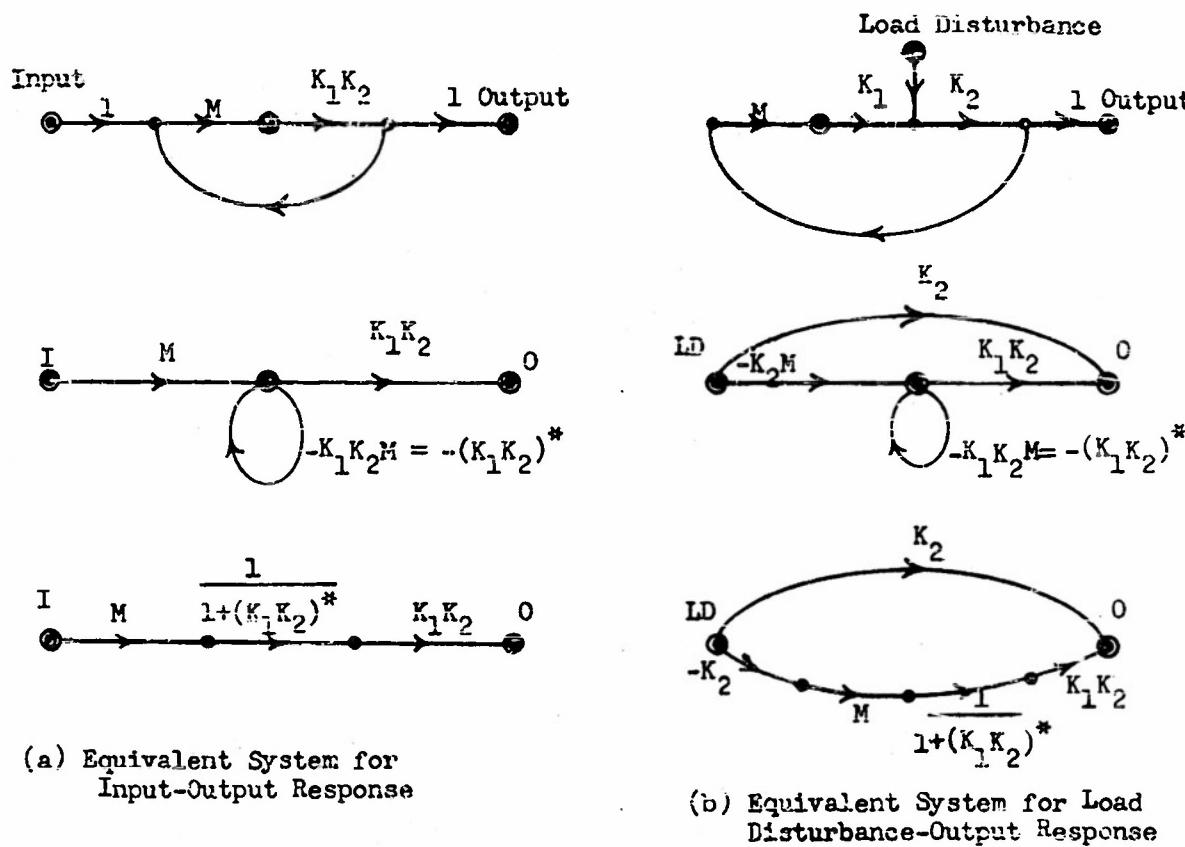
Fig. 32. Reduction of the flow graph of the single-loop servo system.

As an alternative to this procedure we may simply juggle the blocks
and signals in the system block diagram, making full use of linearity.
Although this procedure leads in more general networks to very cumbersome
manipulations, the above system is simple enough to indicate its utility.
It should be understood that nothing new is being added. The following
method is less sophisticated but mechanically more involved than the procedure
for reduction already given to which it is basically equivalent.
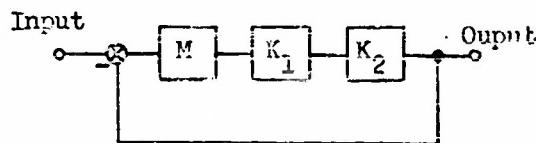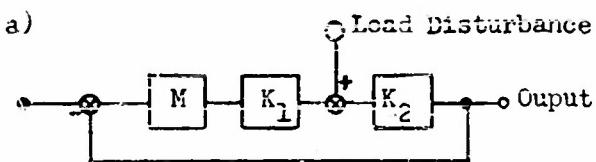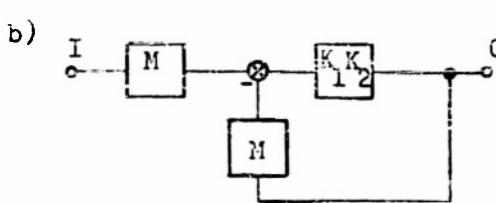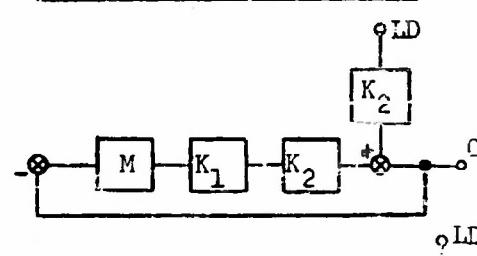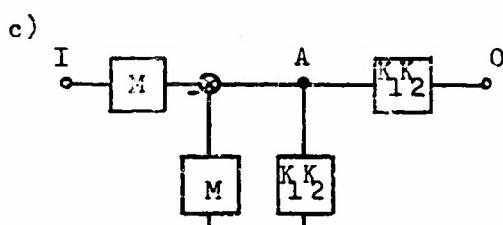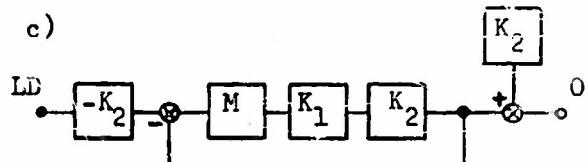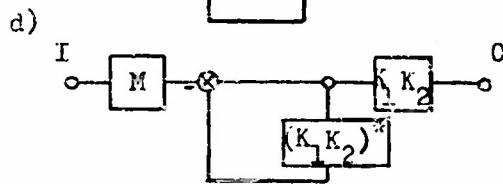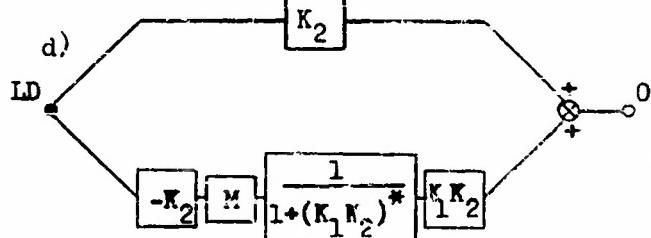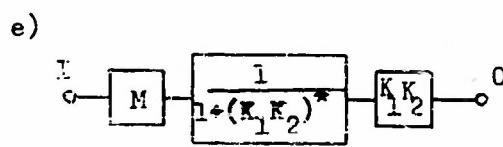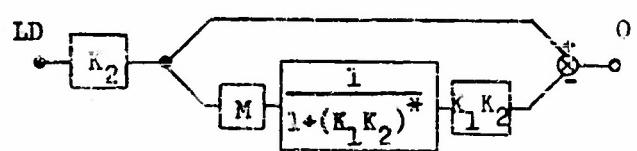
Fig. 33.. Block diagram manipulations
for obtaining input-output
relations for servo system.

Fig. 34. Block diagram manipulations
for obtaining disturbance
responses of servo system.

These block diagram transformations are effected by making use
of the fact that each component block of the system denotes a linear operation
and the fact that the overall operation performed on sampled input signals
by a continuous-signal filter followed by an impulse modulator is just that
performed by a certain discrete signal filter. Referring to Fig. 33, we
see that the step from 33a to 33b results from the linearity of the modulation
process. Because of this linearity it is immaterial whether the fed-back
signal is first subtracted from the input signal and the result sampled or
whether both signals are first sampled and then subtracted. Step 33b to 33c
rests merely on the observation that similar operations (here indicated
by the composite transfer function $K_1 K_2$) performed on the same signal results
in equal signals. Step 33c to 33d is the reduction of the cascade
filter-modulator system to an equivalent discrete signal filter. Note that
at point A the signals are entirely of a sampled nature, since they are
delivered from impulse modulators. Thus we have the branch $K_1 K_2$-M with
discrete input signals which reduces to the equivalent filter with transfer
function $\frac{1}{T} \sum K_1 K_2$ or $(K_1 K_2)^*$ as described in Sec. 1. of this paper.
Finally to go from 33d to 33e we use the well-known reduction of a feedback
loop with open-loop transfer function $(K_1 K_2)^*$ and feed-forward transfer
function of 1.

The transformations involved in Fig. 34 proceed similarly. All
steps depend on linearity; in addition step 34c to 34d makes use of the
overall transformation given in Fig. 33.

If we examine the final result of these manipulations we see that
we are left with series-parallel combinations of the three fundamental
sampled-data system components, modulators, discrete-signal filters, and

continuous-signal filters. The equivalent systems obtained are the same
as these derived by use of the more general reduction procedure. It is now
possible to solve for the response of these networks in a stepwise manner
by means of methods of Sec. 1. The implicit relationships in the original
feedback problem have been reduced to explicit ones.

3. DESCRIPTION OF DESIRABLE SERVO SYSTEM RESPONSE CHARACTERISTICS

The earlier sections of this paper have dealt with analysis of
sampled-data systems. Having thus completed analysis, we now turn to design.
Design is basically different from analysis. Analysis problems usually
have unique solutions. Often there exist no unique design problems for
a practical situation, and even when a unique design problem exists it seldom
has a unique solution. At the beginning of a design problem one has certain
design objectives on one hand and certain physical limitations on the other.
At first neither is clearly defined. The process of design is a cut-and-try
process of realizing the design objectives within the framework of the
physical limitations. Usually design objectives are limited and made more
specific because of knowledge of physical limitations and because of
the designer's limiting himself to a specific class of situations. The
essence of evolving a design technique is to restrict and clarify both
objectives and realizability conditions so that the cut-and-try process
can be a manageable one. Once evolved, a design process has three
component parts: (1) formal statement of objectives, (2) statement of
conditions of physical realizability conditions, (3) an approximation
problem where objectives are realized within the framework of realizability
conditions. The last section of this paper is devoted to the study of design
techniques for sampled-data systems. The first step is to present the
procedure for formal statement of design objectives.

A sampled-data servo is usually designed to have specific response characteristics. The design objectives for sampled-data servos can accordingly be stated in terms of acceptable response characteristics. In short, the responses of a sampled-data servo will probably be acceptable if: (1) its transients are short and adequately damped, (2) disturbances in the feed-forward section are adequately suppressed, (3) errors in following smooth input signals are small, and (4) sampling ripple in the output is small. It turns out that the nature of the transients can be adequately characterized by locating the poles of the overall system transfer function. The quality with which a single-loop servo meets the next two conditions can be approximately determined by use of error coefficients. Transient studies made in Sec. 1 provide all the tools necessary to analyze transient responses and ripple of a system adequately. Accordingly, the response characterizations will be completed by a study of error coefficients.

3.1 Error Coefficients for Sampled-Data Servos

Usually the signals handled by servo systems are smooth, with only infrequent discontinuities. This smoothness of the input signal allows them to be resolved into Taylor's series much more conveniently than they can be resolved into Fourier series or into sets of impulses. Therefore it is often better to characterize a servo system by its error response to a constant, a linear input, a parabolic input, a cubic input, and so on, than it is to describe it in the frequency domain or by its impulse response alone. It should be understood at the outset that this error-coefficient technique is merely another form of superposition which is particularly useful with servos because their inputs are usually very conveniently resolved into power series. Furthermore, servos are most often follow-up devices, and the objective is that they shall follow with small error.

The response accordingly can be judged by knowing the error rather than
by knowing the output. The[8] successive error coefficients of a servo
system are the steady-state values of the error response at $t = 0$ to a
constant, a straight line passing through the origin with unit slope, a
parabola passing through the origin with unit second derivative, a cubic
curve passing through the origin with unit third derivative, and so on.
The great majority of servo systems can be described adequately in terms
of their error coefficients and in terms of the duration of their transients.

As with continuous-data servos, sampled-data servos can also
be described very well in terms of error coefficients. The size of the
error coefficients determines how well a given servo will perform in
many applications. Since the error in a sampled-data servo is sampled
and since all inputs to it are sampled, the error coefficients will be
found in terms of sampled inputs and sampled error. The block diagram which
relates the input to the error may be obtained by the techniques of Sec. 2.
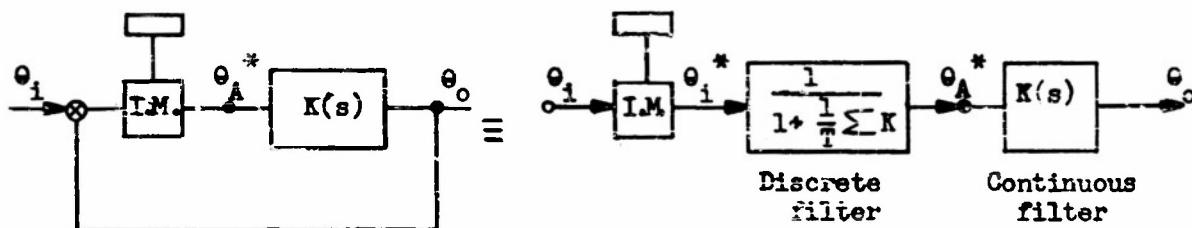

Fig. 35. Equivalent cascaded system of a one-loop sampled-data servo.
The relation between Θ_A^* and Θ_i^* can be obtained by the transfer function,
$$\frac{1}{1+\frac{1}{T}\sum K},$$ of the discrete filter. Knowing the transfer function as a

8 James, Nichols, and Phillips, Theory of Servomechanisms, McGraw-Hill Book Co.,
  New York, 1947, Vol. 25, M.I.T. Rad. Lab. Series.

rational function of ϵ^{-sT}, one can find the unit sample response by expanding the transfer function in a power series of ϵ^{-sT}.

The positional error coefficient is the error arising at $t = 0$ from an infinite string of unit samples which have been applied since $t = -\infty$. This error can be evaluated by applying the superposition summation of Eq. 7 as follows: First expand the discrete-filter transfer function into a power series,

$$\frac{1}{1 + \frac{1}{T}\sum K} = h_0 + h_1 \epsilon^{-sT} + h_2 \epsilon^{-sT} + \text{---} \ h_k \epsilon^{-ksT} + \text{---}. \tag{43}$$

Then apply Eq. 7 for the case where n=0 and Eq. 44 results.

$$\Theta_A^*(0) = \left[h_0 + h_1 + h_2 + \text{---} \ h_k + \text{---} \right] \cdot U_0(t), \text{ where} \tag{44}$$

$U_0(t)$ is a unit impulse at the origin. Thus E_0, the positional error coefficient is expressed by Eq. 45 from the area of the error impulse at $t=0$.

$$E_0 = \text{area of } \Theta_A^*(0) = \sum_{k=0}^{\infty} h_k = \left. \frac{1}{1 + \frac{1}{T}\sum K} \right| \tag{45}$$

$$\epsilon^{-sT} = 1 \text{ or}$$
$$s = 0.$$

To find the velocity error coefficient, convolve the unit sample response with an input which is made up of the samples of a signal of unit slope. For this input, $\Theta_i^*(nT)$ = an impulse with area = nT. In Eq. 46 Eq. 7 is applied to the situation involving the new type of input.

$$\Theta_A^*(0) = \left[0 \times h_0 - T \times h_1 + -2T \cdot h_2 + \text{---} + (-kT)h_k + \text{---} \right] U_0(t), \tag{46}$$

where $U_0(t)$ is a unit impulse at the origin. Thus E_1, the velocity error coefficient, is expressed by Eq. 46 from the area of the error impulse at $t=0$.

$$E_1 = \text{area of } \Theta_A^*(0) = \sum_{k=0}^{\infty} -kTh_k = -\frac{d}{ds}\left(\frac{1}{1+\frac{1}{T}\sum \overline{K}}\right)\Bigg|_{\substack{s=0 \\ \epsilon^{-sT}=1.}} \tag{47}$$

The acceleration error coefficient can be obtained by applying Eq. 7 to the case wherein $\Theta_i^*(nT) =$ an impulse with the area $\frac{1}{2!}(nT)^2$.

$$\Theta_A(0)^* = \frac{1}{2!}\left[(-T)^2 h_1 + (-2T)^2 h_2 + \cdots (-kT)^2 h_k + \cdots\right] U_0(t), \tag{48}$$

where $U_0(t)$ is a unit impulse at the origin. Thus E_2, the acceleration error coefficient, is expressed by Eq. 48 from the area of the error impulse at $t = 0$.

$$E_2 = \text{area of } \Theta_A^*(0) = \sum_{k=0}^{\infty} \frac{1}{2!}(-kT)^2 h_k = \frac{1}{2!}\frac{d^2}{ds^2}\left(\frac{1}{1+\frac{1}{T}\sum K}\right)\Bigg|_{\substack{s=0 \\ \epsilon^{-sT}=1.}} \tag{49}$$

The procedure indicated above can be generalized to apply to all error coefficients as in Eq. 50.

$$E_k = \frac{1}{k!}\frac{d^k}{ds^k}\left(\frac{1}{1+\frac{1}{T}\sum K}\right)\Bigg|_{\substack{s=0 \\ \epsilon^{-sT}=1.}} \tag{50}$$

Note that if $\frac{1}{1+\frac{1}{T}\sum K}$ is expanded in a Taylor's series about the origin in the s-plane, the successive constants in the Taylor's expansion correspond to the error coefficients. If $\frac{1}{1+\frac{1}{T}\sum K}$ is given in terms of a rational function of ϵ^{-sT}, it is usually easy to make a Taylor's series of $\frac{1}{1+\frac{1}{T}\sum K}$ in terms of ϵ^{-sT} about $\epsilon^{-sT} = 1$ and then to expand this series into a Taylor's series in s by expanding $\epsilon^{-sT} - 1$ in a power series in s.

$$\frac{1}{1+\frac{1}{T}\sum K} = \frac{1}{1+K^*} = \alpha_0 + \alpha_1(\epsilon^{-sT}-1) + \alpha_2(\epsilon^{-sT}-1)^2 + \cdots. \tag{51}$$

$$(\epsilon^{-sT}-1) = -sT + \frac{(-sT)^2}{2!} + \frac{(-sT)^3}{3!} + \cdots. \tag{52}$$

Combining Eqs. 51 and 52 yields Eq. 53.

$$\frac{1}{1+K^{\#}} = \alpha'_0 + \alpha_1\left[-sT + \frac{(-sT)^2}{2!} + \frac{(-sT)^3}{3!} + ---\right]$$
$$+ \alpha_2\left[(-sT) + \frac{(-sT)^2}{2!} + \frac{(-sT)^3}{3!} + ---\right]^2 + ---.$$

(53)

From Eq. 53 it can be seen that $E_0 = \alpha_0(-T)^0$, $E_1 = \alpha_1(-T)^1$, $E_2 = (\alpha_2 + \frac{1}{2}\alpha_1)(-T)^2$, etc. Usually the positional error coefficient is zero, and often the velocity error coefficient is also zero. Since we are most interested in the first few error coefficients and particularly in the first non-zero one, the Taylor series in $(\epsilon^{-sT}-1)$, Eq. 51, is usually the most convenient relationship to use. Suppose that $\frac{1}{1+K^{\#}}$ has a kth order zero at $\epsilon^{-sT}=1$. Write $\frac{1}{1+K^{\#}}$ as $(\epsilon^{-sT}-1)^k F(\epsilon^{-sT})$. All of the first $(k-1)$ error coefficients will be zero and E_k will be given by $(-T)^k F(\epsilon^{-sT})\Big|_{\epsilon^{-sT}=1}$. If $F(\epsilon^{-sT})$ is a rational function of ϵ^{-sT}, it can be evaluated by taking the ratio of the product of all the vectors whose tails rest on zeros of F and whose tips rest on the +1 point to the product of all the vectors whose tails lie on the poles of F and whose tips rest on the +1 point. Thus a zero-velocity error servo must have a $\frac{1}{1+\frac{1}{T}\sum K}$ function having a double order zero at $\epsilon^{-sT}=1$. It is interesting to note in passing that the behavior of a sampled-data servo corresponds to the behavior of its transfer functions about $\epsilon^{-sT}=1$ in the same sense that the behavior of a conventional servo corresponds to the behavior of its transfer function about $s=0$.

Having found a procedure for evaluating the error coefficients, one can apply them to evaluate the error resulting from a load torque disturbance much as is done with continuous servos. By methods shown in Sec. 2, a steady load torque disturbance can be seen to cause the same steady error as a velocity input, and consequently low velocity error coefficients insure good suppression of load torque disturbances.

Error coefficients and pole positions of the system function are sufficient to describe the essential response characteristics of the system. Error coefficients, and the nature of the transient response can be obtained from a study of the $\dfrac{1}{1+\frac{1}{T}\sum K}$ function. Accordingly, from now on the response characteristics will be characterized by the constellation of poles and zeros of $\dfrac{1}{1+\frac{1}{T}\sum K}$ in the ϵ^{-sT}-plane. It should be emphasized that this study of the sampled signals does not give a measure of the ripple in the continuous output, but that this measure can easily be obtained in a specific case from an analysis such as was given in Sec. 1. Since the design objectives must be cut to the bone to make a design procedure workable, it seems proper to assume that if the sampled output transients are acceptably smooth and if the output section is low-pass enough, the continuous-transient response will be acceptable. Always after a given design is worked out the ripple can be calculated for specific cases.

3.2 Restrictions Placed on Compensation by Physical Realizability Conditions

The previous section was characterized by a somewhat arbitrary (though sensible) limiting of the desired response description. This section is characterized by a similar limiting of the types of compensation considered. The procedure presented is applicable to a wide variety of problems. In principle it may be extended to fit a much wider range of situations. The

point of view and the analysis technqiue are actually more important than
the formal compensation procedure specifically presented. The reader is
again reminded that to be workable a design procedure must be much less
than completely general. The design procedure given here would serve as
a starting point for the study of a specific design, and as the design
progressed a more elaborate set of procedures could be worked out. As with
continuous servos, practical considerations might make the specific
configuration presented here unworkable. Many of the conditions derive
here would carry over to other situations, so the comments below are more
general than the situation they are applied to. Figure 36 shows the
uncompensated system and the type of compensation proposed. The system

(a) Uncompensated system   (b) Compensated system
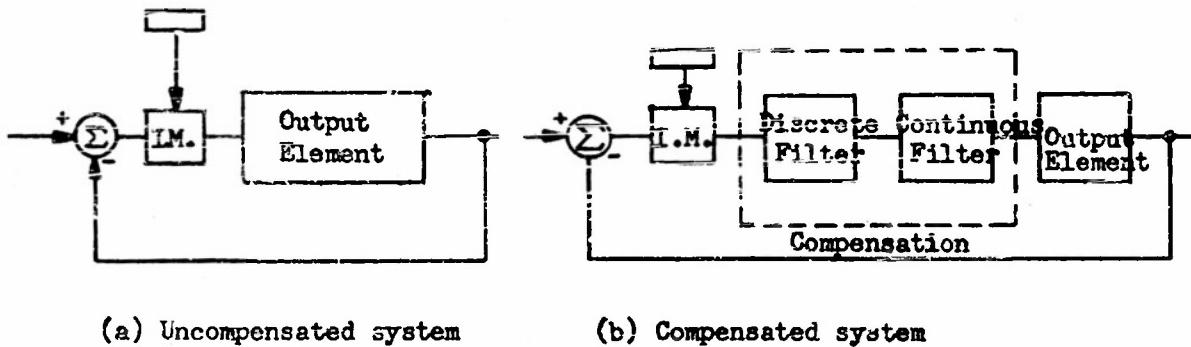
Fig. 36. A simple servo without and with compensation in the forward loop.

has unity feedback, and the compensation filter can be an entirely discrete
filter, an entirely continuous filter, or a combination of the two. The
overall input-to-error transfer function, $\dfrac{1}{1+\frac{1}{T}\sum K}$, is the one used. It
is modified by compensation networks to change it to more acceptable values.
The output element has certain characteristics which can be changed only
at considerable expense. Accordingly they are assumed fixed. The compensation

technique is to select a desirable $\dfrac{1}{1 + \frac{1}{T}\sum K}$ function and then work
backward to the compensating filters. First physical realizability conditions
should be placed on the filter transfer functions. From these filter
realizability conditions, conditions on $\frac{1}{T}\sum K$ are derived.

3.21 Physical Realizability Conditions on Transfer Functions of Discrete and Continuous Filters

The realizability conditions on transfer functions of a single
filter are fairly simple to state. For a discrete-signal filter, the transfer
function must be a rational function in ϵ^{-sT}. The fundamental realizability
condition for any filter is that the response cannot occur before the
input. This condition dictates that the transfer function have no poles
at the origin in the ϵ^{-sT}-plane; it is the only realizability condition
on discrete filters which is fundamental. Usually a discrete filter would
be expected to be stable, though unstable filters can be realized. A
stable discrete filter must have transfer-function poles located only
outside the unit circle centered at the origin of the ϵ^{-sT}-plane. The
fundamental physical realizability condition for continuous filters is
that the response cannot occur before the excitation, and this requirement
is translated into the frequency domain by the Paley-Wiener criterion.
Actually we will deal with transfer functions which are rational functions
of s which all satisfy the Paley-Wiener criterion. For practical reasons,
such as parasitic capacitance in electric filters, it is not feasible to
have transfer functions having more zeros than poles in the finite s-plane.
Accordingly, usable continuous-filter transfer functions will be limited
to those rational functions of s having no more poles than zeros in the
finite s-plane.

Having the conditions of physical realizability compactly stated
for component filters which will be cascaded together to make up the
feed-forward section of a sampled-data servo, the designer must now turn
to the problem of assessing what limitations are placed on the overall
$\frac{1}{T}\sum K$ function by the fixed part of the system.

3.22 Physical Realizability Limitations Placed on Feed-forward Transfer Functions, $\frac{1}{T}\sum K$, by Limitations of Fixed Member and Restriction to Discrete Filter Compensation

Given a fixed output member with a transfer function $K_f(s)$,
one can calculate its sampled equivalent as $\frac{1}{T}\sum_{n=-\infty}^{\infty} K_f(s+jn\Omega)$. If the
discrete filter to be used has a transfer function $D(\epsilon^{-sT})$, then the
overall sampled transfer function for the feed-forward section is
$D(\epsilon^{-sT}) \cdot \frac{1}{T}\sum_{n=-\infty}^{\infty} K_f(s+jn\Omega)$. The physical realizability condition of
the whole sampled feed-forward function is then restricted only to guarantee
that $D(\epsilon^{-sT})$ be realizable or that it have no poles at the origin. In
short, the overall sampled feed-forward transfer function will be realizable
if it has at least as many zeros at the origin of the ϵ^{-sT}-plane as
$\frac{1}{T}\sum_{n=-\infty}^{\infty} K_f(s+jn\Omega)$ does.

3.23 Physical Realizability Limitations Placed on Feed-forward Transfer Functions, $\frac{1}{T}\sum K$, by Limitations on Fixed Member and Restriction to Continuous Filter Compensation

If the designer were given a fixed continuous-signal output
member with a transfer function $K_f(s)$ and if he had selected a sampled
overall feed-forward transfer function $K^*(s) = \frac{1}{T}\sum_{n=-\infty}^{\infty} K(s+jn\Omega)$ he would
find the compensating network as follows:

(1) From $\frac{1}{T}\sum\limits_{n=-\infty}^{\infty} K(s+jn\Omega)$ he would find a suitable $K(s)$. (One

$K^*(s)$ always may correspond to any one of several $K(s)$

functions.)[*]

(2) The transfer function of the compensating network would be

K/K_f.

The problem of this part is to find how $K^*(s)$ is restricted by characteristics

of $K_I(s)$ and the fact that $\dfrac{K}{K_f}$ must be physically realizable. As the next

few paragraphs will show the only conditions on $K^*(s)$ such that K/K_f be

physically realizable are these:

(1) If $K_f(s)$ has an excess of poles over zeros of two or more

$K^*(s)$ must have a single zero at the origin in the ϵ^{-sT}-plane.

(2) $K^*(s)$ must have one less zero than pole in the ϵ^{-sT}-plane

so that it can be expanded in a partial fraction expansion

with no constant term.

Condition (1) results from the fact that $\dfrac{K}{K_f}$ can have no more

zeros than poles in the finite s-plane and accordingly K must have the

same excess in numbers of poles over zeros in the finite s-plane that K_f

has. For K to have two more poles than zeros requires that $K^* = \frac{1}{T}\sum\limits_{n=-\infty}^{\infty} K(s+jn\Omega)$

must have a simple zero at the origin in the ϵ^{-sT}-plane, but if

$K^*(s)$ has a simple zero at the origin, an equivalent unsampled K can be

found having any finite number more poles than zeros. One possible

procedure will be outlined below.

1. Resolve $\frac{1}{T}\sum K$ into a partial fraction expansion. If all

---

[*] The fact that a given sampled transfer function $K^* = \frac{1}{T}\sum\limits_{n=-\infty}^{\infty} K(s+jn\Omega)$ does

not lead uniquely to a $K(s)$ is exactly equivalent to the fact that several
different time functions can all lead to the same set of samples so long
as they are alike only at the sampling points. Two filters having different
$K(s)$ functions will have the same $K^*(s)$ functions if their impulse responses
are alike at the sampling points but different between sampling points.

poles are not simple, treat multiple-order poles as a group of simple poles with small separation. The form of the expansion is given in Eq. 54.

$$K^*(s) = \left[\frac{\alpha_1}{1 - \epsilon^{-(s-a_1)T}} - \frac{1}{2}\alpha_1 \right] + \left[\frac{\alpha_2}{1 - \epsilon^{-(s+a_2)T}} - \frac{1}{2}\alpha_2 \right] + \cdots . \tag{54}$$

2. To each term in the partial fraction expansion associate an unsampled filter transfer function.

$$\alpha_i \left[\frac{1}{1 - ^{-(s-a_i)T}} - \frac{1}{2} \right] \rightarrow \frac{\beta_i^{(o)}}{s-a_i} + \frac{\beta_i^{(-1)}}{s-a_i+j\Omega} + \frac{\beta_i^{(1)}}{s-a_i-j\Omega} + \cdots . \tag{55}$$

The only constraint among the β_i's is that $\beta_i^{(-k)}$ and $\beta_i^{(k)}$ be conjugates and that the sum of all β_i's adds up to α_i. The simplest filter results from using the low-pass term $\beta_i^{(o)}$ only for each term in the partial-fraction expansion of $\frac{1}{T}\sum K$.

If $\frac{1}{T}\sum K$ has a zero at the origin, then $\sum_{i=1}^{n} {}_i = 0$. This result can be seen by adding together the terms in Eq. 54. If $\sum_{i=1}^{n} {}_i = 0$, then

$\sum_{i=1}^{n} \sum_{0}^{m} \beta_i^{(k)} = 0$. If only the low-pass equivalent is used, then $\sum_{i=1}^{n} \beta_i^{(o)} = 0$ or the continuous filter has the sum of all of its residues equal to zero whence it must have two more poles than zeros. The additional freedom in selecting the β's allows one to make $sK(s) \rightarrow 0$ as $\frac{1}{s^2}$ as $s \rightarrow \infty$ so that K can be made to have the denominator three degrees higher than the numerator. By continuing to use the additional freedom in selecting β's, one can make the degree of the denominator of an equivalent K be as many degrees higher than the numerator as he desires. The only cost is a more complicated $K(s)$ function. The procedure to select an equivalent

K with a denominator m degrees higher than the numerator is to select
the β's such that the sum of all $β_i$'s add up to α_i, and further, that
the β's have such values that: the sum of the residues of sK equal zero,
the sum of the residues of s^2K equal zero, and so on including the sum
of the residues of $s^{m-2}K$ equal zero. As the degree of the denominator
exceeds the degree of the numerator by a greater and greater amount, the
number of simultaneous algebraic equations the β's must satisfy increases.[*]

The fact that a given $K^*(s)$ having a zero at the origin in the
ϵ^{-sT}-plane can be realized by a $K(s)$ having a much higher degree denominator
than numerator is physically reasonable. If a $K^*(s)$ function has a single
zero at the origin, the initial sample of the unit-sample response will
be zero. The fact that the degree of the denominator of $K(s)$ is much higher
than the numerator requires that many initial derivatives of the impulse
response of the continuous filter be zero but does not limit what values
it can have at T seconds after the impulse was applied. Accordingly
one would expect the requirement that $\bar{K}^*(s)$ has a zero at the origin if
the degree of $\bar{K}$'s denominator exceeds the degree of its numerator by two
or more and is not surprised that there are no other restrictions on the
sampled transfer function.

The restrictions placed on $\frac{1}{T}\sum K$ by use of discrete-filter
compensation and by continuous-filter compensation have similarities. If
$K_f(s)$ has two more poles than zeros, both types of compensation must lead
to $\frac{1}{T}\sum K$'s which have a single zero at the origin in the ϵ^{-sT}-plane

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[*] Given $K^*(s)$ one can employ still further freedom in going to a $K(s)$ function.
Given any $K(s)$ which yields $K^*(s)$, one can add to it any further transfer
function whose impulse response goes through zero at all the sampling
points. One such zero-sample function is $\dfrac{\beta}{(s+\alpha)^2 + \Omega^2/4}$ where β and
are any two real numbers. See Appendix B for further details.

regardless of whether discrete or continuous-filter compensation are used.

For continuous-filter compensation the number of zeros of $\frac{1}{T}\sum K$ must be one less than the number of its poles (Condition 2) because there is no continuous-signal lumped-parameter filter realization for a $K^*(s)$ function which is a polynomial in ϵ^{-sT}. (It could be realized by delay lines, however.)[*] This restriction does not apply to discrete filters. For discrete filter compensation the number of zeros at the origin in ϵ^{-sT}-plane of $D \cdot \frac{1}{T} K_f$ must be no less than the number of zeros of $\frac{1}{T} K_f$. This restriction does not apply to continuous filters.

In any case, the compensation is simpler if the poles of $\frac{1}{T}\sum K_f$ are included among the poles of $\frac{1}{T}\sum K$ than if $\frac{1}{T}\sum K$ is selected without attention to the poles of $\frac{1}{T}\sum K_f$. Use of a combination of discrete and continuous filters is often useful because it allows one additional freedom in getting $K(s)$ which may then be used for ripple suppression. The use of a combination of discrete and continuous filters results in more relaxed realizability conditions than would be required for either discrete or continuous filters alone. The only condition of realizability on $\frac{1}{T}\sum K$ for compensation by a combination of discrete and continuous filters is that if K_f has two more poles than zeros $\frac{1}{T}\sum K$ must have a zero at the origin in the ϵ^{-sT}-plane. Aside from this restriction, any rational function of ϵ^{-sT} is realizable.

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[*] Note that if $K^*(s)$ includes the transfer function of a clamper then the degree of the numerator of $K^*(s)$ can be equal to the degree of the denominator. Call

$$K(s) = \frac{(1-\epsilon^{-sT})(G(s))}{s} = \left[1-\epsilon^{-sT}\right]\left[\frac{G(s)}{s}\right], \quad K^* = (1-\epsilon^{-sT}) \cdot \frac{1}{T}\sum_{n=-\infty}^{\infty}\frac{G(s+jn\Omega)}{s+jn\Omega}.$$

Given $K^*(s)$, to find $\frac{G(s)}{s}$ one would resolve $\frac{K^*(s)}{1-\epsilon^{-sT}}$ into a partial fraction expansion. Such an expansion would have no constant term if the degree of the numerator of $\frac{K^*(s)}{1-\epsilon^{-sT}}$ was 1 less than the degree of the denominator. This requirement demands that the degree of the numerator of $K^*(s)$ be no greater than the degree of the denominator.

From the description of realizability conditions on the sampled feed-forward transfer function one can deduce conditions of physical realizability on the function $\dfrac{1}{1+\frac{1}{T}\sum K}$.

3.24 Restrictions on $\dfrac{1}{1+\frac{1}{T}\sum K}$ Function from Realizability Conditions on the Compensating Network.

Given the realizability conditions on the feed-forward transfer function, the conditions on $\dfrac{1}{1+\frac{1}{T}\sum K}$ can be derived.

Let:

$$K^*(s) = \frac{1}{T}\sum_{n=-\infty}^{\infty} K(s+jn\Omega) = \frac{P(\epsilon^{-sT})}{Q(\epsilon^{-sT})} . \tag{56}$$

$$\frac{1}{1+\frac{1}{T}\sum K} = \frac{Q}{P+Q} . \tag{57}$$

Given $\dfrac{1}{1+\frac{1}{T}\sum K}$ as a rational function of ϵ^{-sT}, one immediately has Q from the numerator polynomial. Knowing Q and (P+Q) one finds P by finding the difference. P will have a zero at the origin of

$$Q\bigg|_{\epsilon^{-sT}=0} = \Big[P+Q\Big]\bigg|_{\epsilon^{-sT}=0} . \tag{58}$$

The design in terms of $\dfrac{Q}{P+Q}$ is particularly easy because setting the zeros of the fraction at the poles of the sampled fixed function leads to simple compensation networks.

3.3   The Design of Sampled-Data Systems by Approximating to Design Objectives with Realizable Compensation

The design procedure sketched below is somewhat aribtrary and may be modified to adapt it to specific situations. The steps of the design procedure are these:

1. Analyze the fixed parts of the system. Find the position of the poles of $\frac{1}{T}\sum K_f$ in the ϵ^{-sT}-plane and make sure that the sampling frequency, the bandwidth of the servomotor elements, and the bandwidth of expected signals are consistent.

2. Sketch a map of desired poles and zeros of $\dfrac{1}{1+\frac{1}{T}\sum K}$ in the ϵ^{-sT}-plane. Evaluate the error coefficients and make sure if Q is to have a zero at the origin that the product of pole distances from the origin equals in both magnitude and angle the product of zero distances. Whether the system transients are proper is dependent largely upon the location of the dominant poles of $\dfrac{1}{1+\frac{1}{T}\sum K}$.

3. Having selected $\dfrac{1}{1+\frac{1}{T}\sum K}$, solve for $\frac{P}{Q}$ and then select a D and a K which realizes $\frac{P}{Q}$ and at the same time provides proper smoothing of sampling ripple. This process is largely cut-and-try where one tries the simplest realizations first and proceeds to more complicated realizations as necessity demands.

Since the body of techniques for sampled-data servo system design is quite extensive, it is not feasible to catalogue all of them, but rather to indicate that most usual techniques for continuous-data servo system design have counterparts in the sampled-data case and then give a few illustrative examples.

3.31   Design of a Sampled-Data Distance Indicator

A distance measuring device is to be made to indicate both
distance and velocity as continuous quantities. The distance of the
object is provided by a pulse-position modulated signal with a repetition
frequency of one pulse per second. The maximum pulse delay is a small
fraction of a second. The block diagram of the device is shown in Fig. 37
below. The practical problems associated with the design of the system

(a) Block diagram of equipment
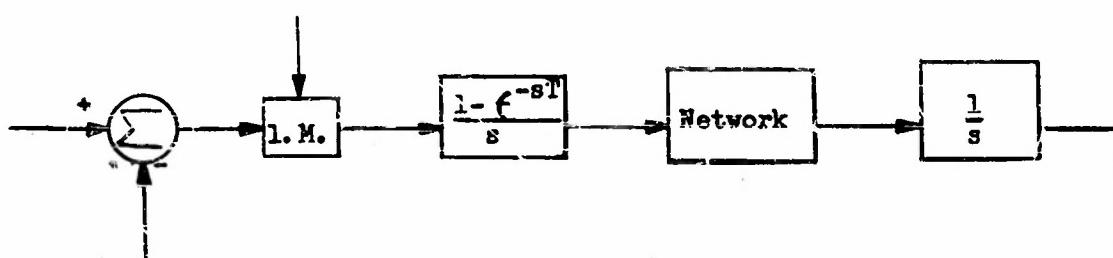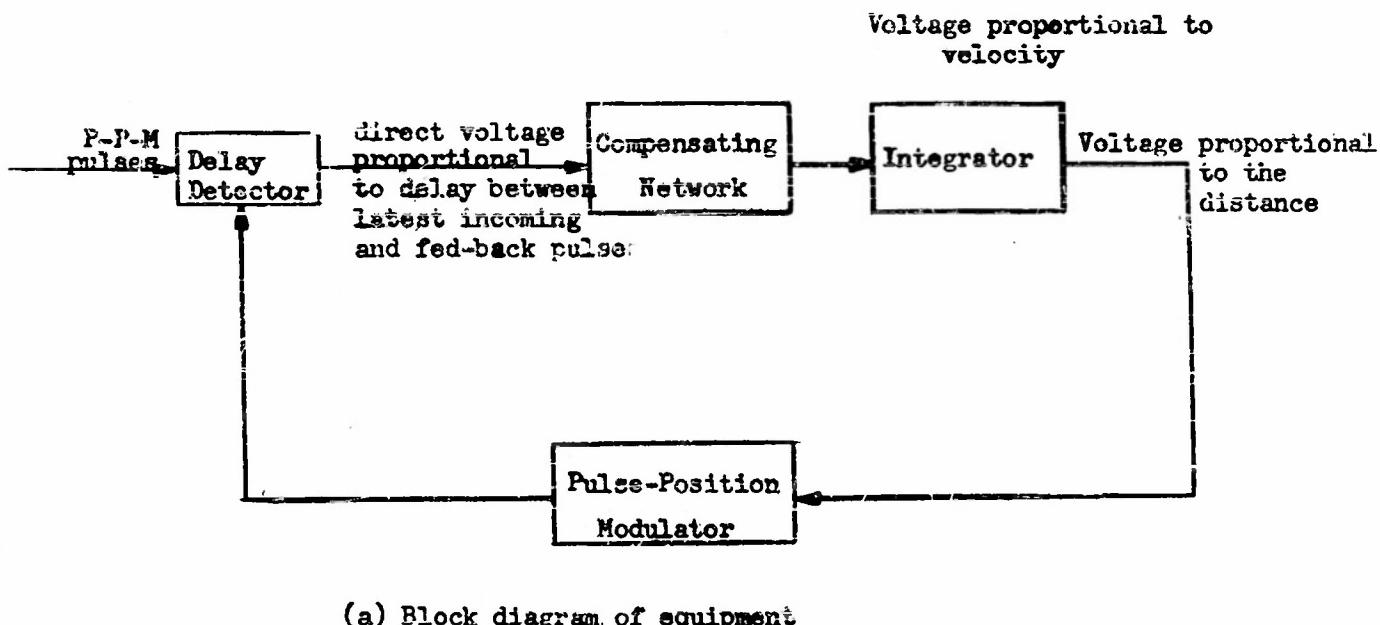
(b) Equivalent diagram for equipment

Fig. 37. The distance measuring device and its equivalent block diagram.

are: (1) The delay detector has a very limited dynamic range and is
linear only around its null point. (2) There is some noise in the actual
distance indicating delay so that while the pure inputs to the device
would correspond to sets of ramps actually there are superposed noise
signals having bandwidths of $\frac{1}{10}$ cycle per second or less. (3) The output
should follow the actual signal as closely as possible. (4) The gain of
the null detector has some drift associated with it. These gain changes
should not adversely affect the system performance.

The first step in the design procedure is to evaluate K_f and K_f^*.

$$\bar{K}_f(s) = \frac{1- \epsilon^{-sT}}{s^2} .$$

$$K_f^*(s) = \frac{(1- \epsilon^{-sT})}{T} \sum_{n=-\infty}^{\infty} \frac{1}{(s+jn\Omega)^2} = \frac{(1- \epsilon^{-sT})T \epsilon^{-sT}}{(1- \epsilon^{-sT})^2} = \frac{T \epsilon^{-sT}}{1- \epsilon^{-sT}} .$$

$$(56)$$

Since K_f^* has a zero at the origin in the ϵ^{-sT}-plane, the overall
feed-forward function must have this same zero. This is the only limitation
on K^*. The selection of a $\frac{1}{1+K^*}$ function can now be made. The selection
of the $\frac{1}{1+K^*}$ function is by no means a unique. The more data the engineer
has on the specific problem, the more intelligently he can use the freedom
in selecting K^* he has gained from knowledge of general realizability
conditions on K^*. In many cases $\frac{1}{1+K^*}$ is the best function to design
in terms of but it is not the only one. In this example we will make a
fairly arbitrary selection of K^* and go through the design before looking
for the alternative possibilities. In order to guarantee a zero velocity
error, $\frac{1}{1+K^*}$ must have a double order zero at the +1 point in the ϵ^{-sT}-plane
for our type of compensation. To make the transient short move all the
poles of the $\frac{1}{1+K^*}$ function to infinity. As a first trial assume

$\dfrac{1}{1+K^*} = \dfrac{(1- \epsilon^{-sT})^2}{1}$. The resulting value of K^* is $\dfrac{2 \, \epsilon^{-sT}(1- \frac{1}{2} \epsilon^{-sT})}{(1- \epsilon^{-sT})^2}$.

Since the denominator of $\dfrac{1}{1+K^*}$ and its numerator have the same value at $\epsilon^{-sT}=0$, the numerator of K^* is guaranteed to have a root at the origin

The constellation of poles and zeros of K^* is plotted in Fig. 38.
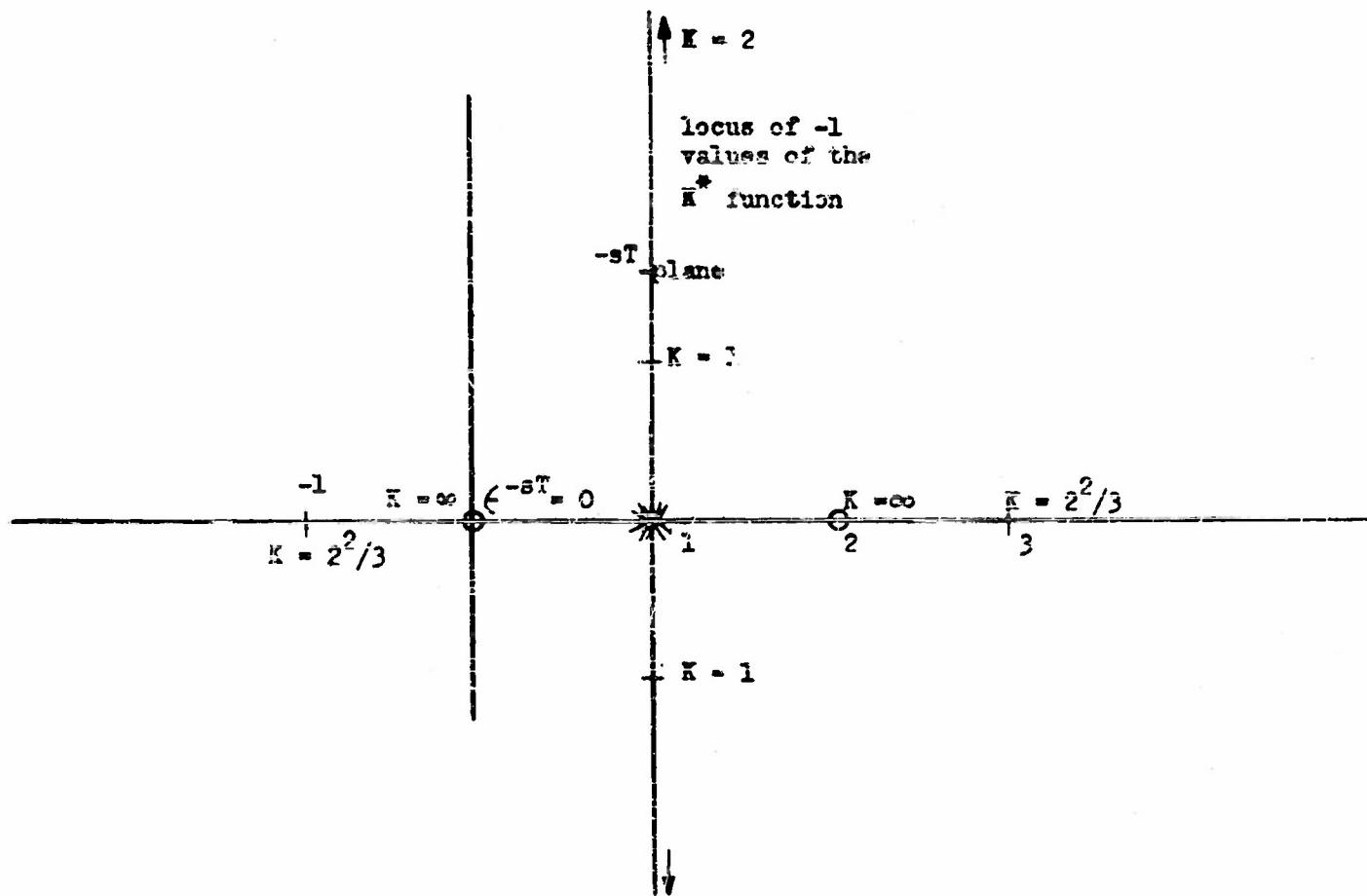
Fig. 38. Poles and zeros of K^* in the ϵ^{-sT}-plane and locus of poles of the system function $\dfrac{\theta_o^*}{\theta_1^*}$ when the gain is changed.

Note if the gain of K is 1 the -1's of K^* occur at $\epsilon^{-sT} = 1 \pm j$; if $K=2$, the -1's of K^* occur at infinity. If $K= 2^2/3$, the -1's of K^* occur at $\epsilon^{-sT} = -1$ and $+3$. This system has a very good transient response, a zero

velocity error coefficient and the system stability has fairly high sensitivity
to changes in loop gain. Note that the _steady-state_ response to steps or
ramps still has a zero error for wide changes in K. Whether one could
tolerate the high sensitivity of the system stability to changes in loop
gain would be dependent upon the variability in gain of the error detector.

The next step in the design is to calculate the transfer function
of the filter needed to obtain the compensation chosen.

$$K^* = \frac{2\,\epsilon^{-sT}(1-\frac{1}{2}\,\epsilon^{-sT})}{(1-\epsilon^{-sT})^2} \quad . \tag{57}$$

Since the clamper has a transfer function $\dfrac{1-\epsilon^{-sT}}{s}$, then the clamper
discrete filter part, $(1-\epsilon^{-sT})$ must be removed and the continuous filter
must give a K_c^* as indicated by Eq. 58.

$$K_c^* = \frac{2\,\epsilon^{-sT}(1-\frac{1}{2}\,\epsilon^{-sT})}{(1-\epsilon^{-sT})^3} \quad . \tag{58}$$

The fact that K_c^* has a third order pole at +1 means that the continuous
filter must have a third order pole at the origin. From table A2 in the
appendix note that $\dfrac{1}{T}\sum\dfrac{1}{(s+jn\Omega)^3} = \dfrac{T^2}{2}\dfrac{\epsilon^{-sT}(1+\epsilon^{-sT})}{(1-\epsilon^{-sT})^3}$. The problem
is to subtract from K_c^* a multiple of the function above such that the
remainder has no third-order pole at $\epsilon^{-sT}=1$.

$$\frac{AT^2}{2}\,\epsilon^{-sT}(1+\epsilon^{-sT}) = 2\,\epsilon^{-sT}(1-\frac{1}{2}\,\epsilon^{-sT})\,\Bigg|_{\epsilon^{-sT}=1} \quad . \tag{59}$$

$$AT^2 = 1,$$
$$A = \frac{1}{T^2}$$

$$K_c^* = \frac{\frac{1}{2}\,\epsilon^{-sT}(1+\epsilon^{-sT})}{(1-\epsilon^{-sT})^3} + \frac{\frac{3}{2}\,\epsilon^{-sT}}{(1-\epsilon^{-sT})^2} \quad . \tag{60}$$

The continuous filter must have the transfer function given by Eq. 61.

$$K_c = \frac{1}{T^2} \times \frac{1}{s^3} + \frac{3}{2Ts^2} = \frac{1}{T^2} \frac{1 + \frac{3T}{2} s}{s^3} . \tag{61}$$

The compensating network transfer function is given by dividing K_c by $\frac{1}{s^2}$ or

$$K \text{ comp.} = \frac{1}{T^2} \frac{1 + \frac{3T}{2} s}{s} . \tag{62}$$
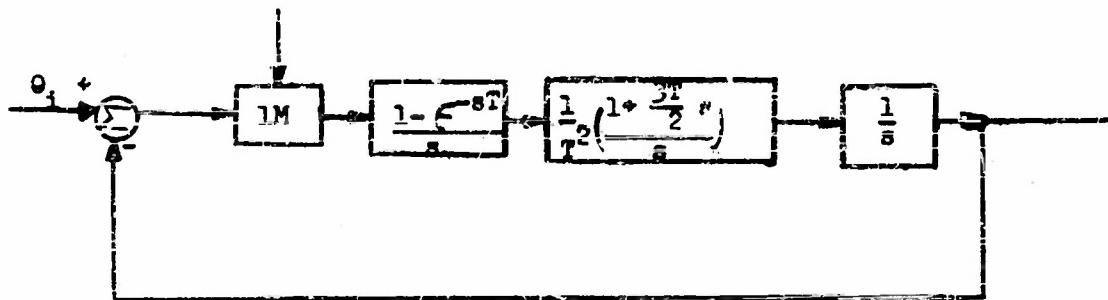
Fig. 39. The proposed system with the compensating network.

Having selected a system function $\dfrac{1}{1 + K^*}$ and having seen how it can be realized, let us now investigate further what the response characteristics of the system are. It is easy to get the sampled impulse response. Call the output of the impulse modulator $\theta_a^*(t)$ (Fig. 38).

$$\frac{\theta_a^*}{\theta_i^*} = \frac{1}{1 + K^*} = 1 - 2\epsilon^{-sT} + \epsilon^{-2sT} . \tag{63}$$

If $\Theta_i^*(t)$ is a single impulse, then the response can be seen as in Fig. 40.
For $\Theta_i^*(t)$, a sampled step, the response is seen to be a step starting at
zero with a sample of 2 at $t = \frac{Z}{2}$ and no error thereafter.
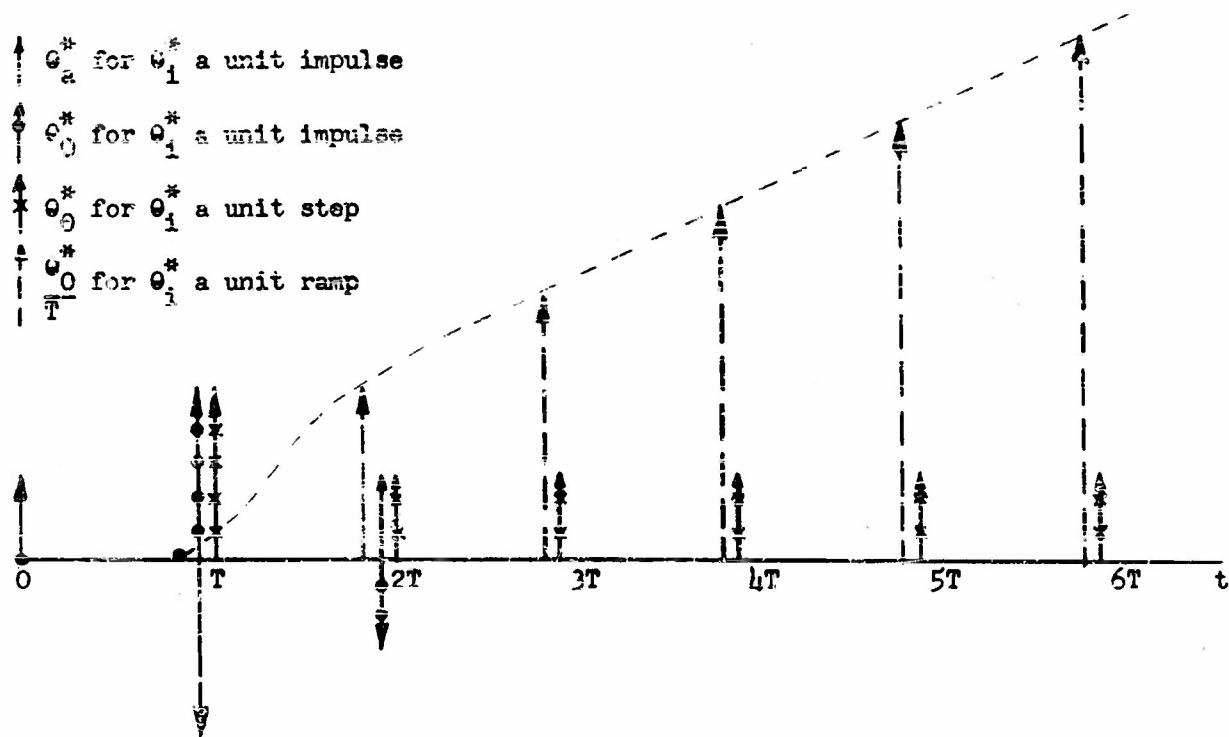

Fig. 40 Sampled outputs for several inputs.

Note that the step response has an overshoot such that the sum of
errors is zero. This overshooting characteristic is a necessary part

of the zero-velocity-error characteristic. The ramp response is plotted
in Fig. 40. The ramp response can be calculated from the step response
by a very simple procedure. Note that the ramp input samples can
be obtained readily from the unit step input samples by delaying them
by 1 sampling interval and then multiplying by T and adding up all
the present and all previous samples. Thus the ramp at $t = T$ equals
T or the T times the sample at $t = 0$ plus all previous ones (or zero).
The ramp sample at $t = 3T$ is T times the sum of step samples at
$t = 2T$, T, and 0 or 3T. Similarly the response to the ramp at $t = 3T$
is found by multiplying by T the sum of the step responses at $t = 2T$,
T, 0. If there is to be no ramp error then the sum of the step
respone samples must equal the sum of the step input samples.

Having a picture of the samples of the ramp response, next
we should try to get a measure of the actual continuous output.
The transform of the actual output when a ramp input is applied is
given by multiplying the transform of the sampled ramp by the
transfer function relating $\theta_c(s)$ to $\theta_i^*(s)$.

$$\theta_0(s) = \frac{T \epsilon^{-sT}}{(1- \epsilon^{-sT})^2} \times (1- \epsilon^{-sT})^2 \times \frac{(1- \epsilon^{-sT})(1+ \frac{3T}{2} s)}{T^2 s^3}. \quad (64)$$

$$\theta_c(s) = \frac{(1- \epsilon^{-sT}) \epsilon^{-sT}(1+ \frac{3T}{2} s)}{Ts^3} . \quad (65)$$

Since $\theta_o(s)$ has no poles except at the origin, it can readily be seen that the only types of long-term response are parabolas, ramps or steps.

$$\theta_o(s) = \epsilon^{-sT} \times \left(\frac{1}{Ts^3} + \frac{3}{2} \frac{1}{s^2}\right) - \epsilon^{-2sT}\left(\frac{1}{Ts^3} + \frac{3}{2} \frac{1}{s^2}\right) \quad (66)$$

The output time function can be evaluated by evaulating the inverse transform of the elements of θ_o. The inverse transform of $\frac{1}{Ts^3}$ is $\frac{1}{2T} t^2$ for t greater than zero. The inverse transform of $\frac{3}{2} \frac{1}{s^2} = \frac{3}{2} t$ for t greater than zero. The inverse transform of $\epsilon^{-sT} \frac{1}{Ts^3} + \frac{3}{2} \frac{1}{s^2}$ is $\frac{1}{2T}(t-T)^2 + \frac{3}{2}(t-T)$ for $t > T$. The inverse transform of $\epsilon^{-2sT}\left(\frac{1}{Ts^3} + \frac{3}{2} \frac{1}{s^2}\right)$ is $-\frac{1}{2T}(t-2T)^2 - \frac{3}{2}(t-2T)$ for $t > 2T$. At all other values of t the response is zero. For $t > 2T$ the total response is

$$\frac{1}{2T}\left[(t-T)^2 - \left[(t-T)-T\right]^2\right] + \frac{3}{2}\left[t-T - t+2T\right] = (t-T) + T. = t.$$

The output curve is plotted in Fig. 41.
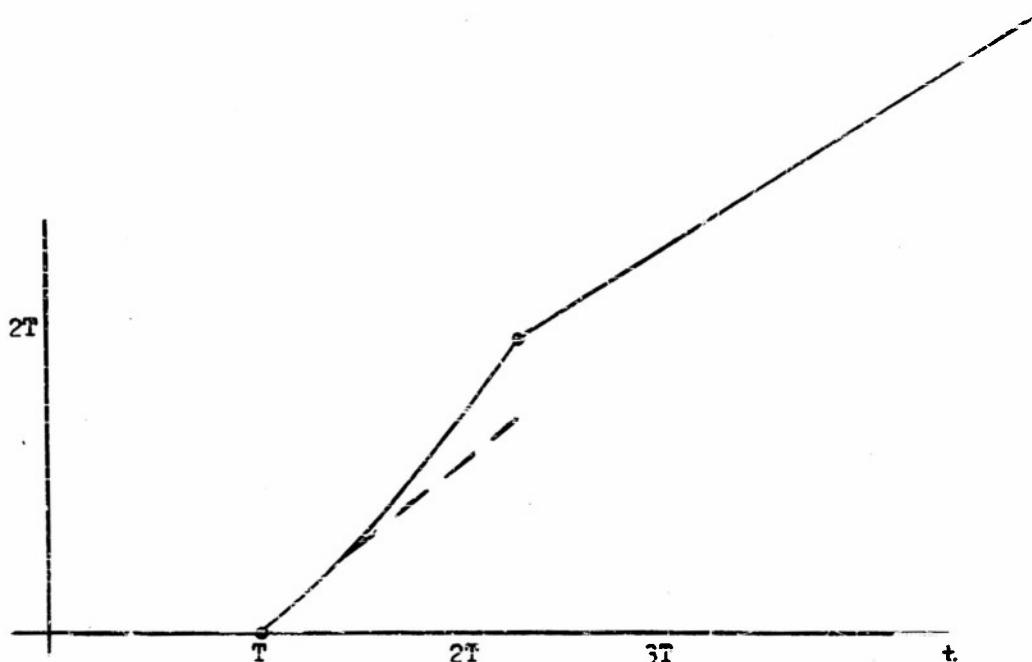
Fig. 41 Response of the system to a ramp input.

Having followed through a trial design we should now observe in
retrospect what was done. In the first place the process illustrated here
was not a "one-shot" synthesis procedure to get an "optimum system." The
process is trial-and-error and what has been done so far is to make one
trial design and evaluate it. The fact that system characteristics are
given compactly in terms of the sampled-input-to-sampled-error transfer
function and that realizability conditions are easy to state means that the
first trial can be made with enough circumspection to be a successful
one. For our example the first trial could be characterized as possessing
very good steady-state characteristics and short transients. The fact
that there is high overshoot to the step response means that the system
would not be satisfactory to handle inputs with many discontinuities
in them. Response to a ramp is certainly smooth and fast. Fig. 38
indicates that the system will not be overly sensitive to gain change.
If discontinuities in the input occur often enough so that the high
overshoot in the step response is objectionable, then the $\dfrac{1}{1+K^*}$ could
be rechosen. The fact that a zero-velocity-error system is specified
implies that the step response must overshoot if it starts off at zero,
but the severity of the overshoot can be diminished by making the over-
shoots last longer. If $\dfrac{1}{1+K^*} = \dfrac{(1-\epsilon^{-sT})^2}{1-\epsilon^{-sT}/2}$, the peak overshoot
to a step can be seen to be $\dfrac{1}{2}$ but the transient lasts for a much longer
time. It should be observed that one has much freedom in the design
procedure but to use it intelligently he must have many practical details
of the system in mind. Since the problem chosen here was merely used
as an example, it will not be carried further.

3.32 Compensation of a Sampled-Data System with a Motor Output

Consider the system pictured in Fig. 42. Suppose the sampling
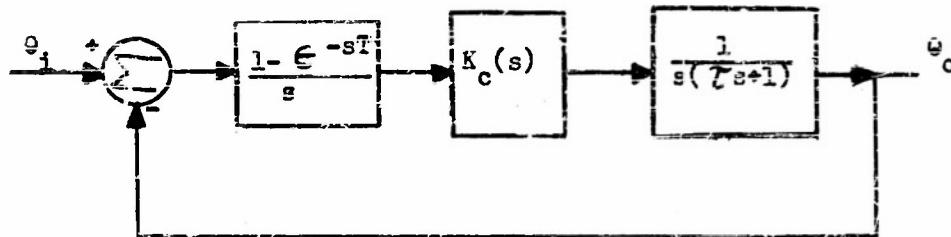
Fig. 42. A unity feedback system with a motor output.

interval is 1 second and that the motor time constant T is 0.721 second,
design a system with zero velocity error, short transients, and one which
requires only a continuous filter for compensation. (Actually one could
prescribe a discrete compensating filter but it would be more expensive
to realize than a passive electric filter would be.) The transfer function
of the feed-forward section is $\left(1-\epsilon^{-sT}\right)\dfrac{1}{s^2(Ts+1)} K_c(s)$. The $(1-\epsilon^{-sT})$
factor is essentially that of a discrete filter, and the fixed continuous-filter
transfer function $\dfrac{1}{s^2(Ts+1)}$ implies that the overall continuous filter
must have three more poles than zeros. The pole at $s= -\dfrac{1}{T}$ implies a pole
of K^* at $\epsilon^{-sT}= 4$. From these restrictions on K^* one can select reasonable
$\dfrac{1}{1+K^*}$ functions. As a start try

$$\frac{1}{1+K^*} = \frac{(1- \epsilon^{-sT})^2(1- \epsilon^{-sT}/4)}{1} . \tag{67}$$

This insures a zero velocity error system with a zero at $\epsilon^{-sT}=4$ to use
the pole of K^* arising from the fixed part of the system. The impulse
response of the system is obtained from multiplying $\dfrac{1}{1+K^*}$ by 1.

$$KG^* = \left(1-2\epsilon^{-sT}+\epsilon^{-2sT}\right)\left(1- \frac{\epsilon^{-sT}}{4}\right) = 1- \frac{9}{4}\epsilon^{-sT}+ \frac{3}{2}\epsilon^{-2sT}- \frac{1}{4}\epsilon^{-3sT}.$$

$$\tag{68}$$

Figure 43 shows the output samples arising from an input which is a sampled step. Again the fact that the system must have overshoots to equal
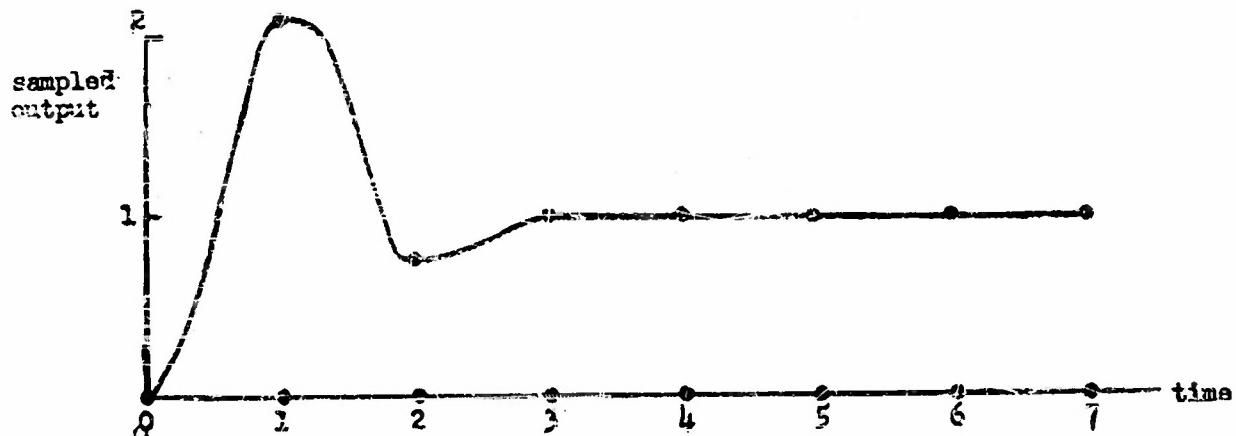
Fig. 43 Response of proposed system to a step input.

undershoots in its step response to have a zero velocity error coefficient and the fact that the transient is short requires high overshoot. Assuming that this transient is acceptable, let us evaluate K^* and eventually $K_c(s)$.

From Eq. 67, K^* can be calculated to be

$$K = \frac{\frac{9}{4}\epsilon^{-sT} - \frac{3}{2}\epsilon^{-2sT} + \frac{1}{4}\epsilon^{-3sT}}{(1-\epsilon^{-sT})^2\,(1-\epsilon^{-sT}/4)}.$$

$$\left[\bar{K}_c(s) \, \frac{1}{s^2(\,s+1)} \right]^* = \frac{\frac{9}{4}\epsilon^{-sT} - \frac{3}{2}\epsilon^{-2sT} + \frac{1}{4}\epsilon^{-3sT}}{(1-\epsilon^{-sT})^2 \, (1-\epsilon^{-sT}/4)} \, . \qquad (70)$$

Given $\left[K_c(s) \, \frac{1}{s^2(\,s+1)} \right]^*$, one is not forced to go uniquely

to $K_c(s) \, \frac{1}{s^2(\,s+1)}$ but can go to a function $K_{c.s.}$ having only poles within

the central strip $- \Omega/2 < \omega < \Omega/2$. To this $K_{c.s.}$ one can add any

"zero-sample" K; that is, one whose impulse response goes through zero

at all sampling points.

Resolve $\left[K_c(s) \, \frac{1}{s^2(\,s+1)} \right]^*$ into partial fractions so as to

be able to identify terms.

$$\frac{\left(\frac{9}{4} - \frac{3}{2}\epsilon^{-sT} + \frac{1}{4}\epsilon^{-2sT}\right)\epsilon^{-sT}}{(1-\epsilon^{-sT})^3 \, (1-\epsilon^{-sT}/4)} = \frac{\frac{2}{3}\epsilon^{-sT}(1+\epsilon^{-sT})}{(1-\epsilon^{-sT})^3} + \frac{\frac{14}{9}\epsilon^{-sT}}{(1-\epsilon^{-sT})^2}$$

$$+ \frac{\frac{1}{27}}{1-\epsilon^{-sT}} - \frac{\frac{1}{27}}{1-\epsilon^{-sT}/4} \, . \qquad (71)$$

The $K_c(s)\dfrac{1}{s^2(\tau s+1)}$ having poles within the strip $-\Omega/2 < \omega < \Omega/2$ can

be identified from Eq. 71 by methods described in Appendix B.

$$K_{c.s.} = \frac{\frac{4}{3}}{s^3} + \frac{\frac{14}{9}}{s^2} + \frac{\frac{1}{27}}{s} - \frac{\frac{1}{27}}{s+1.387} \quad .$$

$$(72)$$

$$= \frac{1.61s^2+3.49s+1.85}{s^3(s+1.387)} \quad .$$

$K_{c.s.}$ alone will not lead to a realizable $K_c(s)\dfrac{1}{s^2(\tau s+1)}$ because it has

only two more poles than zeros. Add to $K_{c.s.}$ a "zero-sample" $\bar{K}$ such that

the combination has 3 more poles than zeros. One such zero-sample K is

given by Eq. 73.

$$\bar{K}_{z.s.} = \frac{j\beta}{s+3+j\pi} + \frac{-j\beta}{s+3-j\pi} = \frac{2\pi\beta}{s^2+6s+9+9.89} = \frac{2\pi\beta}{s^2+6s+18.89}$$

$$(73)$$

Select the value of β such that $\bar{K}_{c.s.} + \bar{K}_{z.s.}$ has 3 more poles

than zeros. For this to be the case $2\pi\beta = -1.61$.

$$\frac{K_c(s)}{s^2(s+1)} = \frac{1.61s^2+3.49s+1.85}{s^3(s+1.387)} + \frac{-1.61}{s^2+6s+18.89}$$

$$(74)$$

$$= \frac{10.915s^3+56.81s^2+77.1s+34.98}{s^3(s+1.387)(s^2+6s+18.89)} \quad .$$

$$\bar{K}_c(s) = \frac{10.915s^3+56.81s^2+77.1s+34.98}{1.387s(s^2+6s+18.89)} \quad . \qquad (75)$$

The $K_c(s)$ is certainly physically realizable. The only trouble with it

is that it is probably more complicated than it really needs to be.

Such complication should have been anticipated because we did not concern
ourselves with realization problems until the end of the design procedure.

The design procedure just discussed selected a desirable $\frac{1}{1+K^*}$
function which was realizable and proceeded to derive the transfer function
of the compensating filter. Though the procedure was workable it gave
too much attention to system response and too little attention to the form
of the compensation. An alternative procedure would be to find what functions
can be added to the fixed $K(s)$ function so as to leave the excess of
poles over zeros at 3 and still provide a more desirable $\frac{1}{1+K^*}$. Actually
by adding to $K(s)$ one would add to K^* and it would be easier to evaluate
the new $1+K^*$ than its reciprocal. For this cut-and-try design no neat
schemes have been worked out for the general case and considerable computation
and plotting is required.

Multiple-loop systems can be designed much as they are in
conventional servos but the procedures will not be illustrated in this
report. The procedure which has been used is to reduce the compensation
problem to an equivalent single-loop problem and to derive conditions upon
the system so that the compensation is realizable. Overall response
characteristics are chosen and the compensation network is derived.

3.4 Summary of Design Procedures

The first problem in designing sampled-data servomechanisms
is the problem of selecting an output member and a basic sampling rate
from the general performance requirements of the system. The output
member is selected on the same basis as the output member would be selected
for a continuous servo. The sampling rate is determined by the bandwidth
required in the system loop. After the fixed parts of the system are
selected, the compensation is chosen. To select the compensation one
adopts a trial and error procedure to approximate the desired response

characteristics in terms of the compensation filters which are physically
realizable. The first part of this section dealt with simple approximate
characterization of the responses of a sampled-data system. The second
part dealt with physical realizability conditions and the last part dealt
with the realizing the response characteristics in terms of the framework
of physical realizability limitations. The last procedure must fundamentally
be cut-and-try and a good procedure must be characterized by short and
simple trial steps.

APPENDIX A

Procedures for Evaluating $K^*(s)$ from $K(s)$ and $\Theta_i^*(s)$ from $\Theta_i(s)$

As has been pointed out earlier, the relation of $K^*(s)$ to $K(s)$ is the same as the relation of $\Theta_i^*(s)$ to $\Theta_i(s)$ though $K(s)^*$ and $K(s)$ are transfer functions and $\Theta_i^*(s)$ and $\Theta_i(s)$ are transforms of time functions. $K^*(s)$ and $K(s)$ may be visualized as being transforms of an impulse response. The problem discussed in this appendix will be the problem of relating $\Theta_i^*(s)$, the Laplace Transform of an impulse-modulated signal, to $\Theta_i(s)$, the Laplace Transform of the unsampled signal. Because of the fact that the relation between $\Theta_i(s)$ and $\Theta_i^*(s)$ is linear, superposition of component Θ_i's leads to superposition of Θ_i^*'s, and it is wise to build up a table of pairs. By using this table one can go from a given Θ_i to Θ_i^* by resolving the Θ_i into component Θ_i's which are available in the table, associating each with its Θ_i^*, and finally superposing Θ_i^*'s. Needless to say, the Θ_i —— Θ_i^* table forms a two-way street and the same sort of procedure can be used for associating a Θ_i^* with a possible Θ_i. The _inverse_ process will be discussed in Appendix B.

Since most Θ_i functions are rational in s, it is easy to resolve Θ_i into a partial fraction expansion. Accordingly, the table should include Θ_i functions of the form $\frac{1}{s+\alpha}$ where α is a complex constant. Table A1 below gives a list of useful transform pairs between continuous and impulse-modulated time functions. Because of the fact that $\frac{1}{s+\alpha}$ corresponds to a function having a discontinuity at t=0 which is a sampling point, there is uncertainty about the Θ_i^* associated with it. The next few paragraphs will explore and remove the confusion arising from this uncertainty..

TABLE A1

Pairs of Transforms of Continuous Time Functions and

Their Impulse-Modualted Equivalents

$\Theta_i(s)$	$\Theta_i^*(s)$
$\dfrac{1}{s+a}$	$\dfrac{1}{1-\epsilon^{-(s+a)T}}$.
$\dfrac{1}{(s+a)^2}$	$\dfrac{T\cdot\epsilon^{-(s+a)T}}{\left[1-\epsilon^{-(s+a)T}\right]^2}$.
$\dfrac{1}{(s+a)^3}$	$\dfrac{T^2\,\epsilon^{-(s+a)T}(1+\epsilon^{-(s+a)T})}{2\left[1-\epsilon^{-(s+a)T}\right]^3}$.
$\dfrac{\partial^n}{\partial s^n}\left(\dfrac{1}{s+\alpha}\right)$	$\dfrac{\partial^n}{\partial s^n}\left(\dfrac{1}{1-\epsilon^{-(s+\alpha)T}}\right)$.

Note: All continuous time functions having transforms rational in s
and having impulse modulated equivalents can be related to their
impulse modulated equivalents by use of the table and partial
fraction expansions. The same is true for the process of going
from the impulse-modulated function to the equivalent time function.

According to the view of the impulse modulator as a sideband generator,

$$\Theta_1^*(s) = \frac{1}{T}\sum_{n=-\infty}^{\infty} \Theta_1(s + jn\Omega), \text{ where}$$

(A0)

$$\Omega = \frac{2\pi}{T}, \text{ and}$$

T = interval between samples.

If $\Theta_1(s) = \frac{1}{s+\alpha}$, the value of $\Theta_1^*(s)$ can be evaluated as

$$\Theta_1^*(s) = \frac{1}{T}\sum_{n=-\infty}^{\infty} \frac{1}{s+\alpha+jn\Omega} = \frac{1}{1-\epsilon^{-(s+\alpha)T}} - \frac{1}{2}.$$

(A1)

Equation (A1) can be verified by a summation process. This result agrees with the one given in Eq. 5 of Section 1.1 in the text. The transform $\frac{1}{s+\alpha}$ corresponds to a string of impulses in the time domain whose successive areas are $\epsilon^{-n\alpha T}$ with a first impulse whose area is $\frac{1}{2}$ because only half of the first sample is caught by an ideal impulse modulator. The value of Θ_1^* can be obtained by adding up the transforms of the individual impulses.

$$\Theta_1^*(s) = -\frac{1}{2} + 1 + \epsilon^{-\alpha T} \cdot \epsilon^{-sT} + \epsilon^{-2\alpha T} \cdot \epsilon^{-2sT} + \cdots = \frac{1}{1-\epsilon^{-(s+\alpha)T}} - \frac{1}{2}$$

(A2)

The equivalence of Eqs. (A1) and (A2) indicates agreement between the "complementary signal" approach and the "shifted impulse" approach. Which of the two one should use depends upon which summation is easier to form. The fact that the time function corresponding to $\frac{1}{s+\alpha}$ has a discontinuity at the origin which is a sampling point and one might not wish to assume that physical equipment would catch only half of the first sample. Accordingly one can advance the wave by shifting it to

the left by a small amount, δ . The first sample caught then would become
1 instead of $\frac{1}{2}$ and performing the approach of Eq. (A2) would yield

$$\Theta_1^*(s) = \frac{1}{1- \epsilon^{-(s+\alpha)T}} \quad . \tag{A3}$$

A check with the approach of Eq. (A1) can be obtained by performing the
summation of all complementary signal components.

$$\Theta_1^*(s) = \lim_{\delta \to 0^+} \frac{1}{T} \sum_{n=-\infty}^{\infty} \frac{\epsilon^{\delta(s+jn\Omega)}}{s+\alpha+jn\Omega} = \frac{1}{1-\epsilon^{-(s+\alpha)T}} \quad . \tag{A4}$$

If none of the first sample were caught, then the summation of
shifted impulses would start with the one which occurred at $t=T$, the transform
of which is $\epsilon^{-sT} \times \epsilon^{-\alpha T}$. The approach of Eq. (A2) would yield

$$\Theta_1^*(s) = \frac{\epsilon^{-(s+\alpha)T}}{1- \epsilon^{-(s+\alpha)T}} = \frac{1}{1-\epsilon^{-(s+\alpha)T}} - 1. \tag{A5}$$

To insure that the sample at $t=0$ is not caught, the time function
could be delayed by δ seconds where δ is a very small number. The
approach of Eq. (A1) can be checked by again performing a summation on
all complementary signal components.

$$\Theta_1^*(s) = \lim_{\delta \to 0^+} \frac{1}{T} \sum_{n=-\infty}^{\infty} \frac{\epsilon^{-\delta(s+jn\Omega)}}{s+\alpha+jn\Omega} = \frac{1}{1-\epsilon^{-(s+\alpha)T}} - 1. \tag{A6}$$

Summarizing the results of Eqs. (A1) through (A6), the relation
between $\frac{1}{s+\alpha}$ and the corresponding Θ_1^* depends upon what assumption is
made about the size of the sample taken at the discontinuity. The choice
made in Table A1 was made arbitrarily and only because of the simplicity
in the resulting formulae. In practical cases, few sampled time functions

really have discontinuities at the origin and accordingly the sum of the
residues is zero.   This fact means that which form is chosen is inconsequential.
As an example, assume $F(s)$ is given by Eq. (A7).

$$F(s) = \frac{a_1}{s+\alpha_1} + \frac{a_2}{s+\alpha_2} + \frac{a_3}{s+\alpha_3} \text{ , where} \tag{A7}$$

$$a_1 + a_2 + a_3 = 0.$$

Make the identification of Eq. (A8).

$$\left(\frac{1}{s+\alpha}\right)^* = \frac{1}{1-\epsilon^{-(s+\alpha)T}} + \beta. \tag{A8}$$

$$F^*(s) = \frac{a_1}{1-\epsilon^{-(s+\alpha_1)T}} + \beta a_1 + \frac{a_2}{1-\epsilon^{-(s+\alpha_2)T}}$$

$$+ \beta a_2 + \frac{a_3}{1-\epsilon^{-(s+\alpha_3)T}} + \beta a_3. \tag{A9}$$

$F^*(s)$ does not involve β since the sum of its coefficients add
to zero.   This fact will always be the case in practical situations because
no physical quantities really are represented by time functions having
discontinuities.

APPENDIX B

Procedures for Evaluating $K(s)$ from $K^*(s)$ and $\Theta_i(s)$ from $\Theta_i^*(s)$

The problem in going from $\Theta_i^*(s)$ to $\Theta_i(s)$ has really one more
complication than the problem of going from $\Theta_i(s)$ to $\Theta_i^*(s)$ because the
continuous function cannot be uniquely defined from its samples. The
procedure arbitrarily adopted here is to select the $\Theta_i(s)$ having its
poles in the strip around the origin in the s-plane. This $\Theta_i(s)$ is
referred to as the "central strip" Θ_i. One can add to this Θ_i^* c.s.
any transform corresponding to a time function whose samples all are zero.
Such a function need not be analytically defined but in practical cases,
particularly in connection with relating sampled and continuous-signal
transfer functions, it can be most conveniently used if the Θ_i having
zero samples is rational in s. Two kinds of "zero sample" Θ_i's have
been used. One kind is made up of functions whose poles lie on the
lines $s = \pm jk\Omega/2$ and whose residues in these poles are imaginary.
Each of these functions will be associated with time functions represented
by rotating vectors having zero real part at the sampling instants.
The other kind is made up of functions having poles separated by $jk\Omega$
in the s-plane and having residues whose sums are zero. As illustrated
in the examples in the report, it is often possible to use the freedom
from non-uniqueness in going from $\Theta_i^*(s)$ to $\Theta_i(s)$ to practical advantage
but full advantages of the use of this freedom have not been explored.